MARCHANT — METHODS

STRAIGHT-LINE INTERPOLATIONS

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Remarks: An oft recurring task in business, engineering, and scientific calculating where factors are obtained from tables, is the determination of tabular values that are in-between the printed ones. As ordinarily done, the process involves several subtractions, a multiplication, a division, and a final addition, and even if a calculator is used there is likelihood of error owing to the necessity of copying intermediate figures to a work sheet and re-setting on the calculator. The majority of interpolating is done by the "straight-line" principle; viz., the unknown variable is assumed to vary during the interval in direct proportion to that of the known. The Marchant by the method presented herein provides means of doing this quickly and without the necessity of intermediate copying. (The Marchant is also ideal for making "curvilinear interpolations" by Lagrangean Coefficients, details of which will be supplied upon request.)

Example: A bond basis book shows a value of 98.8877 on Dec. 31, '38, and 98.7878 on June 30, '39. What is the value on Apr. 23, '39, using 360-day basis? (Total Interval for the known variable is 180 days, and for the unknown, the interval is 113 days from first date.)

Operations: Decimals: Upper Dial 5, Middle Dial 10, Keyboard Dial 5. Use 8 or 10-column Marchant with Upper Green Shift Key down.

1. Set up in Keyboard Dial the Last Value (98.7878) and multiply by the known interval from First Value to Value Desired (113).
2. Clear Keyboard Dial only, and set up First Value (98.8877) and multiply by known interval from Value Desired to Last Value (180 — 113, equals 67) by building Upper Dial to read the Total Interval (180).
3. Clear Keyboard and Upper Dials, set up Total Interval (180) and divide.
   Upper Dial shows Interpolated Value Desired (98.8250).

Note: It will be observed that if the Total Interval is 10, 100, 1000, etc., no division is necessary. Most interpolating in scientific tables is of this class; for example, if the Desired Value is 38/100 of the distance, graphically considered, from First Value to Last Value, it is only necessary to multiply the Last Value by .38 and the First Value by .62 and the accumulated sum of the two products is the Desired Value.

The second multiplication may be expedited by multiplying First Value by '1' and then reverse multiplying by .38 which is easily proved correct by its reducing the .38 of the first multiplication to ciphers.

Submitted by
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MARCHANT — METHODS

INVERSE STRAIGHT-LINE INTERPOLATION — DIRECT METHOD

REMARKS: *This is opposite of the direct interpolation as described in Marchant Method MM-52. particularly in the footnote thereof. It is a frequent task in computing from tables.*

EXAMPLE: A bond basis book shows at a given date a yield of 6.60% for bonds bought at 90.3095 and a yield of 6.50% if bought at 91.3781. The bonds were bought at 90.8750. What is the yield?

OPERATIONS: Decimals; Upper Dial 7 and 0, Middle Dial 12 and 5, Keyboard Dial 5. Use 8 column Marchant. If model M is used, have Non-Shift Key down.

(1) With carriage in 1st position, set up in Keyboard Dial the highest value (91.3781) and add.

(2) Similarly, set up the lowest value (90.3095) and reverse multiply by "1".

(3) Shift carriage to 8th position and reverse multiply by "1".

(4) Change Keyboard Dial to read Intermediate Value (90.8750) and add.

(5) Set up in Keyboard Dial the value that is in the Middle Dial at right (1.0686), clear Upper Dial* and divide.

The ratio that the difference between Intermediate Value and Low Value bears to the difference between High Value and Low Value appears in Upper Dial; viz., .5292-, which is interpreted in this case as .05292% of yield because the total difference is 0.1% of yield.

(6) Inspection of problem shows this amount is to be subtracted from 6.60, which may be done by any usual means. One that gives proof control in this instance is:

(a) Clear Keyboard and Middle Dials. Shift to 8th position. Set up in Keyboard Dial the amount from which subtraction is to be made (6.6). Depress Add Bar, and then depress Subtract Bar.

(b) Set up the total rate difference (6.6 - 6.5, or 0.1) in Keyboard Dial, and with Upper Green Shift Key down reverse multiply by the first four significant figures of the amount that appears in the Upper Dial.

Interpolated Yield appears in Middle Dial (6.5471%).

NOTE: *It should not have more significant figures than that of the original values.*

(*) *Not necessary if model has automatic Upper Dial clearance prior to division.*

MARCHANT — SILENT SPEED — METHODS

INVERSE OR DIRECT STRAIGHT LINE INTERPOLATION - BUILD-UP METHOD

REMARKS: *This method should be studied in connection with MM 91. It is sometimes of use in problems of this kind when D models are used, particularly when tables for various intermediate values are to be built up.*

EXAMPLE:

Yield 6.60%	Value 90.3095
" 6.50%	" 91.3781

What is yield at 90.8750?

OPERATIONS: Decimals: Upper Dial 7 and 0, Middle Dial 12 and 5, Keyboard Dial 5. Use 8 column model D or M.

(1) With carriage in 1st position set up in Keyboard Dial the highest value (91.3781) and add.

(2) Similarly set up the lowest value (90.3095) and reverse multiply by "1."

(3) Shift carriage to extreme right and add.

(4) Shift carriage to extreme left and copy amount that is at the right end of the Middle Dial to Keyboard Dial and reverse multiply by "1."

(5) Shift carriage to extreme right, clear Upper Dial and multiply by such an amount as will build up* the Middle Dial to the intermediate value (90.8750).

Ratio that (90.8750 - 90.3095) bears to total interval (91.3781 - 90.3095) appears in Upper Dial (.5292-).

This amount is multiplied by total interval (.1) and subtracted from 6.60 by method of MM 91 if desired, producing desired yield (6.5471%).

NOTE: Either the Upper Dial or the Middle Dial amounts may be the unknown value, so this setting is useful for both inverse or direct interpolation when producing tables in which straight-line proportionality is satisfactory.

(*) *The "build-up" process referred to in Operation No. 5 is to start with carriage as far to the right as possible so that multiplying by from "1" to "9" will cause the amount in the Middle Dial to increase to an amount that is greater than that to which you are building up. With carriage in the specified position (in this case 7), hold down X Bar (or successively depress keys of Automatic Multiplier Keyboard on M models) until the amount in the Middle Dial is just enough greater than the desired amount that a touch of the Short Cut Bar (or a reverse multiplication by "1" on M models) will cause the amount to become less than the desired value. Next shift carriage one position to the left and again multiply until the amount is greater than the desired value. Touch Short-Cut Bar until the amount is just reduced below the desired value. Repeat this process until the Upper Dial shows the number of places desired, the value in the Middle Dial meanwhile more and more closely approaching the desired value.*

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M A R C H A N T   M E T H O D S

CURVILINEAR INTERPOLATION BY THE METHOD OF CONSTANT SECOND DIFFERENCES

EXEMPLIFIED BY COMPUTING TABLE OF CAPACITY OF CURVED BOTTOM TANK AT INCH OF DEPTH SOUNDINGS

REMARKS: In much engineering and scientific work, where straight line interpolation is not regarded as sufficiently accurate, a satisfactory trend of tabular values may be obtained by considering Second Differences as constant for the sub-divisions of the Intervals between calculated values of the table.

EXAMPLE: The calculated volume of a tank with outwardly flaring curved sides, at various depths in even feet is as below:

		First Diff (d')	Second Diff (d")	Avg. d"
7.0 ft	197.87 bbls		3.08	
		33.92		2.85
6.0	163.95		2.62	
		31.30		2.36
5.0	132.65		2.09	
		29.21		1.87
4.0	103.44		1.64	
		27.57		1.47
3.0	75.87		1.29	
		26.28		1.17
2.0	49.59		1.04	
		25.24		.97
1.0	24.35		.89	
		24.35		.83
0	0		.76	

Italics

It is desired to know the volume at intermediate depths at one inch intervals.

OPERATIONS: Decimals; Upper Dial 5 and 0; Middle Dial 14, 9, and 0. Keyboard Dial 9 and 4. Use 10 col. Marchant (with Upper Green Shift Key down on M models.

(1) Compute First and Second Differences (d' and d") by Marchant Method MM-100 and tabulate them as shown in italics above.

(2) Extrapolate d" for 0 ft. and 7.0 ft. as shown in script numerals. This may be done by calculating from 3rd differences or by plotting a few values of d" toward each end and extending their curve to the end points (in this case 0.0 and 7.0 respectively).

(3) Compute the Average d" in each Interval; e.g., the Avg. d" for 6.0 to 7.0 ft. is ½(3.08 + 2.62), or 2.85, etc. and copy these at right of table.

CALCULATION OF FIRST INCREMENTS AND CONSTANT SECOND DIFFERENCES IN EACH TABLE INTERVAL

OUTLINE: <u>First</u> <u>Increment</u> in each Interval between computed values is:

(a) $$\frac{d'}{h} - \frac{\text{Avg. } d''}{h^2} \times \frac{h-1}{2}$$

in which d' = First Difference of Interval (See preceding page)
Avg. d" = Avg. Second Difference during Interval
h = Number of increments in Interval

For machine calculation, the above formula is transposed to:

(b) $$\frac{1}{h^2}\left[d'h - \text{Avg. } d''\left(\frac{h-1}{2}\right)\right]$$

Constant Second Diff. for h increments per Interval is:

(c) $$\frac{\text{Avg. } d''}{h^2}$$

For interpolation of volume between 0.0 and 1.0 ft., d' = 24.35, Avg. d" = .83, h = 12 (12 in. = 1 ft.), and similarly for other Intervals (See preceding page)

For calculation of right-hand factor of Formula (b) above, proceed as follows for all Intervals of table:

(1) Set up d' (24.35) at 4th Keyboard Dial decimal and multiply by h(12) at 5th Upper Dial decimal.

(2) Clear Upper and Middle Dials, and set up Avg. d" (.83) at 4th Keyboard Dial decimal and reverse multiply at 5th Upper Dial decimal by $\frac{1}{2}(h-1)$, or 5.5.

Upper Dial shows ...994.50000(complement of 5.5)

Avg. d" $d'h - \text{Avt. } d''\left(\frac{h-1}{2}\right)$, or 287.635 appears at 9th Middle Dial decimal

(3) Similarly obtain this amount for all other Intervals of the table, recording them in Col. A. on Work Sheet at top of page 3.

(4) Set up in Keyboard Dial at 9th decimal $1/h^2$ (reciprocal of 144 = .006944444) as constant, and multiply by amounts in Col. A, producing First Increments as in Col. B of Work Sheet.

WORK SHEET

Argument	A	B	C
		First Increments	Constant 2nd Diff.
7.0 ft			
	391.365	2.7178	.0198
6.			
	362.620	2.5182	.0164
5.			
	340.235	2.3627	.0130
4.			
	322.755	2.2414	.0102
3.			
	308.925	2.1455	.0081
2.			
	297.545	2.0663	.0067
1.			
	287.635	1.9975	.0058
0			

(5) Without clearing Keyboard Dial, continue to multiply by the respective Avg. d" values (.83, .97, 1.17, etc.) thus producing the values in Col. C of table of Work Sheet.

INTEGRATION FOR TABULAR VALUES

(6) With carriage in 1st position, set up in Keyboard Dial at 4th decimal the First Increment of Interval (0 to 1.0 ft.) (1.9975) and add, clear Upper Dial.

(7) Likewise set up First Increment (1.9975) at 9th and corresponding Constant Second Difference (.0058) at 4th decimal Keyboard Dial
Keyboard Dial reads 1.99750.0058
and multiply by "1" at "0" Upper Dial decimal
Record 1.9975 as Table Value for 0 ft. 1 in.

(8) Change Keyboard Dial at left so it reads same as Middle Dial at right (2.0033) and multiply by "1"
Record 4.0008 as Table Value for 0 ft. 2 in.

(9) Repeat step 8 for all values up to 1 ft., 0 in. developing table below:

1 ft	0 in.	24.3528	rounded	off	as	24.35
	11 "	22.2915	"	"	"	22.29
	10 "	20.2360	"	"	"	20.24
	9 "	18.1863	"	"	"	18.19
	8 "	16.1424	"	"	"	16.14
	7 "	14.1043	"	"	"	14.10
	6 "	12.0720	"	"	"	12.07
	5 "	10.0455	"	"	"	10.05
	4 "	8.0248	"	"	"	8.02
	3 "	6.0099	"	"	"	6.01
	2 "	4.0008	"	"	"	4.00
	1 "	1.9975	"	"	"	2.00
	0					

Upper Dial will show 12, equaling h, as control, though the approach to 24.35 shows completion of steps in First Interval.

(10) Clear Keyboard and Upper Dial, set up in Keyboard Dial such an amount as when subtracted will leave only the computed value to start next Interval (24.35) in Middle Dial at 9th decimal, and subtract.

(11) Set up in Keyboard Dial at 4th decimal the First Increment for Interval 1.0 to 2.0 ft. (2.0663), and add. Clear Upper Dial.

(12) Likewise set up First Increment (2.0663) at 9th and corresponding Constant Second Difference (.0067) at 4th Keyboard Dial decimals and proceed as in steps 7 to 9 to develop table values between 1 ft. and 2 ft.

(13) Repeat steps 10, 11, and 12 for the remaining Intervals, thus completing the table between 2 ft. and 7 ft.

NOTE A: The example shows d' and d" as positive and table is ascending. The method is just as suitable for negative differences or with d' and d" of opposite sign. If Constant Second Difference is opposite in sign from First Increment, set up complement of Constant Second Difference, preceding by 9's and reducing right-hand digit of First Increment by "1". If tabular values are descending (i.e., d' is minus) reverse multiply in the integration steps.

NOTE B: A quick check of correctness of First Increment and Constant Second Difference is that they satisfy the following:

$$\text{Interval Step} = h \times d' + \tfrac{1}{2}h(h-1)d''$$

or in this case

$$24.35 \neq 12 \times 1.9975 + 66 \times .0058 = 24.3528$$

which is as satisfactory as can be obtained with places available.

NOTE C: Rounding off of right-hand digit of Constant Second Difference causes a possible maximum error of its value of, in this instance, .0005. By integrations this is magnified to $\frac{1}{2}(h-1)(h-2)$ times .0005 at table value h-1, or in this case, 55 × .0005 = .0275. This explains why values obtained by integration to 4 places at right of decimal are rounded off to 2 places.

NOTE D: The use of the Upper Dial as a control of count in the integration is most helpful if it be desired to go immediately to any specific intermediate value. The Upper Dial always records the particular tabular value as numbered from point of origin of the Interval.

MARCHANT — SILENT SPEED — METHODS

DIRECT INTERPOLATION AND SUB-TABULATION
(IF FOURTH DIFFERENCES DO NOT EXCEED 1000)

EXPLANATORY APPENDIX TO MARCHANT METHOD MM-189

A NOTE ON OBTAINING 4TH DIFFERENCES FOR USE WITH "COMRIE THROW-BACK" IN EXAMPLE IV

Reference was made in the second paragraph of the "Remarks" section, Page 1, of Marchant Method MM-189, to the fact that in sub-tabulation it is not necessary to obtain third and fourth differences, except at infrequent intervals, and then only in order to obtain their general range as a guide to the selection of method or as a means of obtaining the 4th difference correction of Example IV.

Inasmuch as a 4th difference must be known before the "4th difference correction" can be determined, it might appear that the statement is inconsistent, because obviously 4th differences will normally vary somewhat from interval to interval.

Actually, however, in ordinary computing practice, it will be found that the 4th difference correction generally can be obtained without the necessity of completely tabulating the 3rd and 4th differences. This is because the large majority of functions which are tabulated to the number of places used in ordinary computing, - rarely more than 7 places - will have no great variation in 4th differences; that is to say, a small 5th difference.

By following the procedure below, the tabulation of 4th differences for every interval may usually be avoided.

A plan that does this is to obtain 4th differences at about every fifth or tenth interval and observe their trend, plotting them graphically and obtaining the 4th differences for the intermediate intervals from the curve so drawn.

This will ordinarily give quite as accurate a 4th difference as would actual differencing at each interval, because the graphical method eliminates the error forced into the 4th difference due to rounding up of the right-hand digit of the tabulated function. Such round-ups can affect the right-hand digit of the 4th difference by as much as 8.

In considering the precision of this approximation, it is to be noted that in the computation of the interpolates only "0.184 x 4th difference" is used. This is additional justification for the procedure of eliminating 4th differencing for every interval when the work falls within the class of Example IV.

MARCHANT — SILENT SPEED — METHODS

HANSEN-AHLBERG METHOD FOR OBTAINING PARABOLIC TRENDS

Remarks: Statisticians who extend second-degree curves may readily do so by the procedure herein. This application to cases of statistical trends was brought to our attention by Mr. Raymond Ahlberg, Statistician, Denver, Colo. Similar procedures have been used for interpolation by integration of constant differences (see Marchant Method MM-152).

Outline: The second degree curve is characterized by having a constant second difference. Advantage is taken of this as the basis for the method. The curves may have any of several forms. Examples are

(1) $Y = a + bX + cX^2$ (2) $\text{Log } Y = a + bX + cX^2$
(3) $Y = a + b/X + c/X^2$ (4) $Y = 1/(a + bX + cX^2)$

The example herein is in the form of (1). If (2) applies, it is only necessary to obtain anti-logs of the log Y's that appear in the Marchant. If (3) applies, it is put in the form $X^2Y = aX^2 + bX + c$. The Marchant then gives the values of X^2Y, which when divided by the X^2's gives Y. If (4) applies, it is put in the form $1/Y = a + bX + cX^2$. The Marchant then gives values of 1/Y, the reciprocals of which are the desired Y's.

Example: Given $Y = 7.2131 - .5114X + .3044X^2$, obtained from a least squares analysis. It is desired to tabulate the trend by intervals of 0.1 from $X = 2.0$

Preliminary: Compute four adjacent values of Y in the neighborhood of $X = 2.0$ and tabulate with differences, a check of correctness being the constancy of the second difference; thus,

X	Y	1st diff.	2nd diff.
1.8	7.278836		
		.061488	
1.9	7.340324		.006088
		.067576	
2.0	7.407900		.006088
		.073664	
2.1	7.481564		

Decimals: Upper Dial 1, Middle Dial 10 & 6, Keyboard Dial 10 & 6. Non-Shift Key down on any 10-column M model.

(1) By suitable means, obtain initial entries as follows: Upper Dial 2.0; Middle Dial 7.408 at 10th decimal and .073664 at 6th decimal; Keyboard Dial .074 (rounded .073664) at 10th decimal and constant 2nd difference .006088 at 6th decimal. These are starting values and are always set up in this pattern when signs of both differences are plus (see Note B).

(2) With carriage in 1st position, depress No. 1 Key of Single Row Keyboard. Y for 2.1 (7.482) appears at left of Middle Dial and the new 1st difference (.079752) appears at right. The Keyboard Dial at 10th Decimal is then changed so it reads .080 (the rounded value of .079752). See Note A.

(3) Depress No. 1 key of Single Row Keyboard. Y for 2.2 (7.562) appears at left of Middle Dial and the new 1st difference (.085840) appears at right. The Keyboard Dial at 10th Decimal is then changed so it reads .086 (the rounded value of .085840).

(4) Repeat Step 3 for succeeding values, the Upper Dial showing values of X.

NOTE A: Constant 2nd diff. should be set up as nearly exact as possible. Rounding that space limitations require should be in 1st diffs. and Y's.

NOTE B: If Y's increase but 2nd diff. is negative, set it in complementary form, bridge with 9's and proceed as herein.

If Y's decrease but 2nd diff. is positive, have Manual Counter Control toward operator, and depress Reverse Bar prior to depressing No. 1 key.

If Y's decrease and 2nd diff. is negative, invert the table; i.e., start from the smallest Y. Then, proceed exactly as outlined in the above method except have Manual Counter Control toward the operator.

Submitted by Garland McWhirter
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A SHORT METHOD FOR EVALUATING DETERMINANTS AND SOLVING SYSTEMS OF LINEAR EQUATIONS WITH REAL OR COMPLEX COEFFICIENTS

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FOREWORD
by
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By permission of the author and publishers, we are privileged to issue this highly significant and valuable paper as a part of our Marchant Methods series.

In addition to the applications mentioned in the author's introduction, examples similar to No. 4 on page 4 are frequently found in the field of Statistical Method.

It will be noted that each element of the calculation is obtained as a result of one continuous calculator operation comprising a series of positive or negative summations of products, with or without final division. This being the case, it is obvious that calculators selected for this work should preferably conform to the following specifications:

(1) Multiplication and division, either positive or negative, with accumulation when desired.
(2) Complete capacity carry-over in both product and counting registers.
(3) Straight-line figure proof of entry of multipliers, multiplicands, dividends and divisors; and particularly that both multiplier and multiplicand shall be visible for permitting check of correctness of entry of the multiplication factors at each step of the continuous process.
(4) Positive decimal control, assuring no uncertainty in the entry of amounts with reference to decimal or in the pointing off of the intermediate and final products or quotients.

The within paper, as published, included a Mathematical Appendix containing the mathematical foundation of the method, and also consideration of complex equations whose symmetrical elements are conjugate. This appendix is not reprinted herein.

A SHORT METHOD FOR EVALUATING DETERMINANTS AND SOLVING SYSTEMS OF LINEAR EQUATIONS WITH REAL OR COMPLEX COEFFICIENTS

Prescott D. Crout

1. *INTRODUCTION.* The purpose of this paper is to describe *without proof* a short method for solving arbitrary systems of linear algebraic equations, and evaluating determinants, the quantities involved being either real or complex. The cases considered are:

(1) Arbitrary systems with real coefficients, which occur in obtaining stresses in structures, in solving systems of linear differential equations with constant coefficients (transient problems), etc.
(2) Symmetrical systems with real coefficients, which occur with direct current networks, undamped vibration, deflections in structures, least square processes, Ritz' method, etc.
(3) Symmetrical systems with complex coefficients, which occur with alternating current networks, and forced vibration with dissipation.
(4) Arbitrary systems with complex coefficients, which occur in certain vibration problems involving gyroscopic action.
(5) Systems involving two sets of variables, which occur when the currents in a network are to be found for a variety of impressed voltages, also in the approximate solution of integral equations arising in electric field problems.

(OVER)

The work of solving a system of equations (or evaluating a determinant) is largely concentrated in the determination of an "auxiliary matrix", and is roughly half that required by a matrix multiplication. The process is particularly adapted for use with a computing machine, for each element is determined by one continuous machine operation (sum of products with or without a final division). The setting down of this matrix and of the final solution is the only writing required by the process. The work involved is cut almost in half if the given equations (or determinant) is symmetrical, as very often happens. A "check column" can be carried along if desired.

The amount of work required to obtain a solution is considerably less than that required by Gauss' method[1], even when there is symmetry and the coefficients are real, in which case Gauss' method has been considerably refined by Doolittle[2]. (Gauss' method is much shorter than a solution by determinants.)

The method as given is applicable to m equations in n unknowns, there being no restriction on the rank of the matrix of the coefficients.

2. *DESCRIPTION OF THE METHOD.* Let the given system of equations be specified by its *given matrix*[3], thus:

	X_1	X_2	X_3	X_4	=
(1)	12.1719	27.3941	1.9827	7.3757	6.6355
	8.1163	23.3385	9.8397	4.9474	6.1304
	3.0706	13.5434	15.5973	7.5172	4.6921
	3.0581	3.1510	6.9841	13.1984	2.5393

the first equation being $12.1719x_1 + 27.3941x_2 + 1.9827x_3 + 7.3757x_4 = 6.6355$. The solution requires the formation of one matrix and a set of final results; thus we have an *auxiliary matrix:*

	X_1	X_2	X_3	X_4	=
	12.1719	2.2506	.16289	.60596	.54515
(2)	8.1163	5.0720	1.6793	.0057629	.33632
	3.0706	6.6327	3.9585	1.4193	.19891
	3.0581	-3.7316	12.7526	-6.7332	.060806

and a *final matrix:*

(3)
$$X_1 = .15942$$
$$X_2 = .14687$$
$$X_3 = .11261$$
$$X_4 = .060806$$

The procedure for obtaining the auxiliary matrix from the given matrix is contained in the following rules.

(1) The various numbers, or *elements,* are determined in the following order: elements of first column, then elements of first row to right of first column; elements of second column below first row, then elements of second row to right of second column; elements of third column below second row, then elements of third row to right of third column; and so on until all elements are determined.

(2) The first column is identical with the first column of the given matrix. Each element of the first row except the first is obtained by dividing the corresponding element of the given matrix by that first element.

(3) Each element on or below the principal diagonal is equal to the corresponding element of the given matrix minus the sum of those products of elements in its row and corresponding elements in its column (in the auxiliary matrix) which involve only previously computed elements.

(4) Each element to the right of the principal diagonal is given by a calculation which differs from 3) only in that there is a final division by its diagonal element (in the auxiliary matrix)[3].

(Continued)

1 "NUMERISCHES RECHNEN", BY RUNGE, P. 65.

2 "PRACTICAL LEAST SQUARES", BY LELAND, P. 40.

3 A MATRIX IS A RECTANGULAR ARRAY OF NUMBERS, OR ELEMENTS. THOSE ELEMENTS WHICH HAVE THE SAME ROW AND COLUMN INDEX FORM THE PRINCIPAL DIAGONAL, WHICH SLOPES DOWN TO THE RIGHT STARTING WITH THE ELEMENT IN THE UPPER LEFT CORNER. THE DIAGONAL ELEMENT OF ANY ELEMENT TO THE RIGHT OF THE PRINCIPAL DIAGONAL IS THAT ELEMENT OF THIS DIAGONAL WHICH LIES IN THE SAME ROW AS THE GIVEN ELEMENT. THE DIAGONAL ELEMENT OF ANY ELEMENT BELOW THE PRINCIPAL DIAGONAL IS THAT ELEMENT OF THIS DIAGONAL WHICH LIES IN THE SAME COLUMN AS THE GIVEN ELEMENT.

As examples we have the following typical calculations made in obtaining (2), the letters R and C representing the words "row" and "column", respectively.[4]

R1C3	.16289	$= 1.9827 \div 12.1719$
R2C2	5.0720	$= 23.3385 - 8.1163 \cdot 2.2506$
R4C2	−3.7316	$= 3.1510 - 3.0581 \cdot 2.2506$
R2C5	.33632	$= (6.1304 - 8.1163 \cdot .54515) \div 5.0720$
R3C3	3.9585	$= 15.5973 - 3.0706 \cdot .16289 - 6.6327 \cdot 1.6793$
R4C3	12.7526	$= 6.9841 - 3.0581 \cdot .16289 + 3.7316 \cdot 1.6793$
R3C4	1.4193	$= (7.5172 - 3.0706 \cdot .60596 - 6.6327 \cdot .0057629) \div 3.9585$
R4C4	−6.7332	$= 13.1984 - 3.0581 \cdot .60596 + 3.7316 \cdot .0057629 - 12.7526 \cdot 1.4193$
R4C5	.060806	$= (2.5393 - 3.0581 \cdot .54515 + 3.7316 \cdot .33632 - 12.7526 \cdot .19891) \div (-6.7332)$

Since a modern computing machine gives in one continuous operation a sum or difference of products with or without a final division, we see that *each element of the auxiliary matrix is given by a single machine operation.*[5]

The procedure for obtaining the one-columned final matrix from the auxiliary matrix is contained in the following rules.

(1) The elements are determined in the following order: last, next to last, second from last, third from last, etc.

(2) The last element is equal to the corresponding element in the last column of the auxiliary matrix.

(3) Each element is equal to the corresponding element of the last column of the auxiliary matrix minus the sum of those products of elements in its row in the auxiliary matrix and corresponding elements in its column in the final matrix which involve only previously computed elements.

We see that in forming products only those elements of the auxiliary matrix are used which lie to the right of the principal diagonal and to the left of the last column. The calculations made in obtaining (3) are:

R3C1	.11261	$= .19891 - 1.4193 \cdot .060806$
R2C1	.14687	$= .33632 - 1.6793 \cdot .11261 - .0057629 \cdot .060806$
R1C1	.15942	$= .54515 - 2.2506 \cdot .14687 - .16289 \cdot .11261 - .60596 \cdot .060806$

It may be noted that *each element of the final matrix is given by a single machine operation.*

It is not necessary but is strongly recommended that the values of the unknowns, which compose the final matrix, be substituted in each of the given equations, the result being a number of checks equal to the number of equations. Since the satisfaction of these checks guarantees the correctness of the solution, *it is not necessary to check the calculations which gave the auxiliary matrix and the final matrix*[6]. The first of the four checks obtained from (1) and (3) is:

$$12.1719 \cdot .15942 + 27.3941 \cdot .14687 + 1.9827 \cdot .11261 + 7.3757 \cdot .060806 = 6.6355$$

Evidently *each check requires but one machine operation.*

The above method is applicable as described to n equations in n unknowns. The only writing involved is that required in recording the auxiliary matrix and the final matrix.

(OVER)

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4 ALL NUMERICAL DATA (EXCEPT WHERE EXACT) WAS OBTAINED BY OMITTING A CERTAIN NUMBER OF DECIMAL PLACES FROM THE CORRESPONDING ORIGINAL DATA, WHICH WAS OBTAINED USING A TEN BANK COMPUTING MACHINE RETAINING TEN DECIMAL PLACES THROUGHOUT. THIS APPLIES TO THE GIVEN EQUATIONS AS WELL AS TO OTHER DATA, FOR THE PROBLEMS USED HERE AS ILLUSTRATIONS WERE, FOR THE MOST PART, PREVIOUSLY SOLVED FOR OTHER PURPOSES.

5 IN CARRYING OUT SUCH AN OPERATION IT IS NOT NECESSARY TO FORESEE WHETHER THE RESULT IS POSITIVE OR NEGATIVE.

6 SINCE IN SOLVING A GIVEN SET OF EQUATIONS ONLY A LIMITED NUMBER OF DECIMAL PLACES CAN BE CARRIED, THE QUESTION ARISES AS TO HOW MUCH THE ERROR ACCUMULATED DURING THE COURSE OF THE CALCULATIONS AFFECTS THE FINAL RESULTS. AN ANSWER TO THIS QUESTION CANNOT BE GIVEN, SINCE IT DEPENDS UPON WHETHER OR NOT THE DETERMINANT OF THE SYSTEM IS ON THE POINT OF VANISHING (OR, MORE GENERALLY, UPON WHETHER THE MATRIX IS ON THE POINT OF SHIFTING RANK). NEVERTHELESS IT WILL BE MENTIONED THAT IN SOLVING EACH OF SIX SYSTEMS OF FOUR EQUATIONS IN FOUR UNKNOWNS USING A TEN BANK COMPUTING MACHINE, THE FINAL CHECKS SHOWED THE GIVEN EQUATIONS TO BE SATISFIED EXCEPT FOR ERRORS NUMERICALLY LESS THAN THREE UNITS IN THE NINTH DIGIT. ALSO IT IS KNOWN (FROM A SEPARATE INVESTIGATION) THAT THE VALUES (10) OF THE UNKNOWNS OF THE COMPLEX SYSTEM (8) COMPUTED WITH A TEN BANK MACHINE CONTAIN ERRORS EACH OF WHICH IS NUMERICALLY SMALLER THAN FOUR UNITS IN THE TENTH DIGIT. ALTHOUGH NO GENERAL STATEMENT CAN BE MADE, IT IS BELIEVED THAT IN THE USUAL CASE THE ACCURACY TENDS TO REMAIN HIGH DURING THE COURSE OF THE CALCULATIONS.

3. *SYSTEMS HAVING SYMMETRICAL COEFFICIENTS.* If there is symmetry, the work of computing the auxiliary matrix is cut almost in half by the fact that *if the coefficients of the unknowns (or the elements of the given matrix) are symmetrical about the principal diagonal; each element of the auxiliary matrix below the principal diagonal gives, if divided by its diagonal element, the symmetrically opposite element above this diagonal.*[3] Elements below the principal diagonal of the auxiliary matrix are thus obtained as by-products of calculations made in determining elements above this diagonal.

As an example the symmetrical set of equations:

	X_1	X_2	X_3	X_4	=
(4)	245.95	768.49	233.08	261.13	40.384
	768.49	2665.85	880.54	915.95	90.887
	233.08	880.54	688.91	458.17	6.5783
	261.13	915.95	458.17	652.85	24.471

has the auxiliary matrix:

	X_1	X_2	X_3	X_4	=
(5)	245.95	3.1246	.94769	1.0617	.16420
	768.49	264.63	.57536	.37800	-.13338
	233.08	152.25	380.42	.40259	-.029928
	261.13	100.03	153.15	276.13	-.0017367

and the final matrix:

$$
(6) \qquad \begin{aligned} X_1 &= .55591 \\ X_2 &= -.11591 \\ X_3 &= -.029229 \\ X_4 &= -.0017367 \end{aligned}
$$

In the auxiliary matrix the element in row 3 and column 4 is:

$$
\frac{(458.17 - 233.08 \cdot 1.0617 - 152.25 \cdot .37800)}{380.42} = \frac{153.15}{380.42} = .40259
$$

the numerator 153.15 being recorded in the symmetrically opposite position before the final division by the diagonal element 380.42 is carried out.[4] The final matrix is obtained in the usual manner.

4. *EVALUATION OF DETERMINANTS.* It is true that *the value of a determinant is equal to the product of the elements which form the principal diagonal of the auxiliary matrix.* The auxiliary matrix is computed as described in Section 2, the elements of the determinant being treated in the same manner as the coefficients of the x's before. The auxiliary matrix is square, since there is no column corresponding to the last column in (2); also, there is no final matrix.

As an example, the determinant composed of the first four columns of (1) has an auxiliary matrix composed of the first four columns of (2); hence the value of the determinant is (12.1719) (5.0720) (3.9585) (-6.7332) = - 1645.4.

If the determinant is symmetrical about the principal diagonal, the work of evaluating it is cut almost in half because of the smaller amount of work required in computing the auxiliary matrix. (See Section 3.) For example, the determinant composed of the first four columns of (4) has an auxiliary matrix composed of the first four columns of (5). The value of the determinant is hence (245.95) (264.63) (380.42) (276.13) = $6.8368 \cdot 10^9$.[4]

5. *CONTINUOUS CHECK ON CALCULATIONS.* If desired, a "check column" may be written at the right of the given matrix, each element of this column being the sum of the elements of the corresponding row in the matrix. This column is now treated in exactly the same manner as the last column of the given matrix, the calculations being carried along with those for the other columns, and the result being the addition of corresponding "check columns" to the auxiliary matrix and the final matrix. The check columns thus obtained for (1), (2), and (3) are, respectively,[4]

(7)	55.560	4.5646		1.1594
	52.372	3.0214	and	1.1469
	44.421	2.6182		1.1126
	28.931	1.0608		1.0608

These columns provide checks at all stages of the computation, because:

(1) In the auxiliary matrix any element in the check column is equal to one plus the sum of the other elements in its row which lie to the right of the principal diagonal.

(Continued)

(2) In the final matrix any element in the check column is equal to one plus the sum of the other elements in its row.

For example noting (2), (3), and (7), two of the checks are:

$$1 + 1.6793 + .0057629 + .33632 = 3.0214$$
$$1 + .11261 = 1.1126$$

The above statements are true and the procedure the same for any number of equations and unknowns; also, for the evaluation of determinants, in which, however, only Statement 1 is applicable.

6. *IMPROVEMENT IN ACCURACY.* Since the number of decimal places in the computations is limited, the values obtained for the unknowns are in general not exact. However, if they are placed in the given equations and the differences between the two sides are obtained, and if these differences are then inserted in place of the right hand sides of the given equations, the resulting equations have as their solution the corrections to the values first obtained. Noting that the above differences are obtained in applying the final checks (see Section 2), and that the auxiliary matrix for the modified equations is the same as that for the original equations except for the last column, it follows that *if the column of the differences*[7] *obtained in applying the final checks be annexed to the given matrix and then treated in the same manner as the last column, the corresponding column obtained in the final matrix is composed of the required corrections.*

Since the problem of solving the modified equations is similar to the original problem, the above process may be repeated; thus the final checks on the corrections give data for another column in the given matrix, which leads to a column in the final matrix composed of corrections to the first corrections, etc. In the usual case each application of this process increases the number of significant figures in the results by approximately the same number obtained with the original solution, the data in the given equations being considered exact.

7. *SYSTEMS OF EQUATIONS HAVING COMPLEX COEFFICIENTS.* Since the proofs which establish the above method do not require the quantities involved to be real, the method is applicable to complex equations. The only question is that of whether the required calculations can be easily performed on a computing machine.

Since $(a + jb)(c + jd) = (ac - bd) + j(ad + bc)$ we see that the real and imaginary parts of a sum of products of complex numbers are each a sum of products of real numbers, given by one machine operation. It follows that *each element of the final matrix, and each element of the auxiliary matrix which lies on or below the principal diagonal can be obtained by two machine operations one of which gives the real part, the other, the imaginary part.* It also follows that *each check run on the final matrix consists of two machine operations.*

Each element of the auxiliary matrix which lies to the right of the principal diagonal is given by a sum of products with a final division. The sum is first computed and recorded, after which the final division by the diagonal element $A = a + jb$ is carried out as a separate *multiplication* by $1/A = a/(a^2 + b^2) - jb/(a^2 + b^2)$. Since the best form for recording the supplementary data depends upon how much of it there is, we shall consider separately the following two cases.

Case 1 – *SYMMETRICAL EQUATIONS.* Noting Section 3 we see that *if the coefficients in the given equations are symmetrical about the principal diagonal, the sum required for computing any element of the auxiliary matrix above the principal diagonal is already recorded as the symmetrically opposite element below this diagonal.* It follows that only the last column of the auxiliary matrix requires separately recorded sums, which may be written in a *right supplementary column.* Data on the final multipliers 1/A may be placed in *two left supplementary columns.* (See (9).) In determining the auxiliary matrix the elements in any row which lie in the supplementary columns are determined just before those in that row to the right of the principal diagonal.

As an example the given set of four equations:

	X_1	X_2	X_3	X_4	=
	84.276 71.184j	-91.342 10.741j	32.463 -12.142j	-45.417 -32.984j	6558.50 -185.00j
(8)	-91.342 10.741j	45.374 52.741j	-21.212 74.342j	80.414 -20.732j	-8780.75 670.75j
	32.463 -12.142j	-21.212 74.342j	72.713 -20.221j	37.642 -77.814j	-3463.25 -3535.41j
	-45.417 -32.984j	80.414 -20.732j	37.642 -77.814j	-20.114 -29.741j	-4268.87 1449.04j

(OVER)

7 IN RECORDING THE COMPUTED VALUES OF THESE DIFFERENCES, AS MANY NON-VANISHING DIGITS SHOULD BE KEPT AS CAN BE HANDLED IN THE SUBSEQUENT MACHINE CALCULATIONS.

where the imaginary part of each element is written below the real part for compactness, has the auxiliary matrix:

	$a^2 + b^2$	1/A	X_1	X_2	X_3	X_4	=	RS
	12170.	.0069251	84.276	-.56973	.15379	-.50745	44.336	
		-.0058493 j	71.184 j	.60867 j	-.27397 j	.037241 j	-39.644 j	
	13101.	-.000009789	-91.342	-.1282	.41654	-.10413	-29.887	-5156.8
(9)		-.0087368 j	10.741 j	114.458 j	.087841 j	-.30098 j	45.088 j	-3426.6 j
	7119.4	.0111588	32.463	-10.107	79.444	.75944	-29.404	-2574.2
		.0039928 j	-12.142 j	47.665 j	-28.426 j	-.77633 j	- 8.3789 j	170.19 j
	3284.5	-.00050088	-45.417	34.462	38.265	-1.6452	-51.713	1369.5
		-.0174415 j	-32.984 j	-11.880 j	-83.263 j	57.287 j	-22.421 j	-2925.6 j

where the supplementary columns are partitioned off by vertical lines, and the final matrix:[6]

	X_1	=	20.471
			10.782 j
	X_2	=	-42.652
(10)			37.913 j
	X_3	=	27.275
			-31.498 j
	X_4	=	-51.713
			-22.421 j

The following are typical calculations made in obtaining the above solution. In each case the part computed is specified by "real" or "imag.", after which the letters R, C, LS, and RS are used in place of "row", "column", "left supplementary", and "right supplementary", respectively. In the auxiliary matrix:[4]

Real, R3C3 79.444 = 72.713 - 32.463·.15379 + 12.142·.27397 + 10.107·.41654 + 47.665·.087841

Imag., R4C3 -83.263 = -77.814 - 45.417·.27397 + 32.984·.15379 - 34.462·.087841 + 11.880·.41654

Real, R3CRS -2574.2 = -3463.25 - 32.463·44.336 + 12.142·39.644 - 10.107·29.887 + 47.665·45.088

R3CLS1 7119.4 = 79.444·79.444 + 28.426·28.426

Imag., R3CLS2 .0039928 = 28.426 ÷ 7119.4

Real, R3C4 .75944 = .0111588·38.265 + .0039928·83.263

Imag., R3C5 -8.3789 = .0111588·170.19 - .0039928·2574.2.

In the final matrix:

Real R2C1 -42.652 = -29.887 - .41654·27.275 - .087841·31.498 - .10413·51.713 + .30098·22.421

Case 2 – *NON-SYMMETRICAL EQUATIONS.* In the general case where there is no symmetry, and a sum must be recorded for every element of the auxiliary matrix to the right of the principal diagonal, the supplementary data may be recorded in a *supplementary matrix* having the same number of rows and columns as the auxiliary matrix. Each element to the right of the principal diagonal is the sum required in computing the corresponding element of the auxiliary matrix; the diagonal elements are the final multipliers 1/A; and values of $a^2 + b^2$ may be placed in a left supplementary column.

Since to obtain any element of the auxiliary matrix or the final matrix a real and imaginary part must be obtained, and since the sums involved have twice as many terms as they would have in the case of real coefficients, we see that *the work involved in solving a set of equations with complex coefficients is a little more than four times that required in solving a similar set with real coefficients.*

The procedures just described under Case 1 and Case 2 can be used to obtain the square auxiliary matrix required in evaluating a determinant with complex elements. (See Section 4.)

A check column can be carried along if desired, the entries being complex. (See Section 5.)

8. *EQUATIONS INVOLVING TWO SETS OF VARIABLES.* If two sets of variables are involved, the above method may be used to obtain one set in terms of the other. For example the given set of equations:[4]

	X_1	X_2	X_3	X_4	=	Y_1	Y_2	Y_3	Y_4
	12.172	27.394	1.983	7.376		6.636	0	0	0
(11)	8.116	23.339	9.840	4.947		0	6.636	0	0
	3.071	13.543	15.597	7.517		0	0	6.636	0
	3.058	3.151	6.984	13.198		0	0	0	6.636

(Continued)

of which the first is $12.172x_1 + 27.394x_2 + 1.983x_3 + 7.376x_4 = 6.636y_1$, has the auxiliary matrix:

	X_1	X_2	X_3	X_4	=	Y_1	Y_2	Y_3	Y_4
	12.172	2.251	.1629	.6060		.5452	0	0	0
(12)	8.116	5.072	1.679	.00576		-.8724	1.308	0	0
	3.071	6.633	3.958	1.419		1.039	-2.192	1.676	0
	3.058	-3.732	12.753	-6.733		2.699	-4.877	3.175	-.9855

and the final matrix:

			Y_1	Y_2	Y_3	Y_4
	X_1	=	-9.187	17.053	-12.117	5.643
(13)	X_2	=	3.800	-6.607	4.734	-2.343
	X_3	=	-2.791	4.730	- 2.830	1.399
	X_4	=	2.699	-4.877	3.175	- .9855

from which we see that $x_1 = -9.187y_1 + 17.053y_2 - 12.117y_3 + 5.643y_4$, etc. Each y column of (11) is treated in the same manner as the last column of (1), the result being a corresponding column in the final matrix. Four final checks on each of these is obtained by equating the corresponding y to 1 and the other y's to 0, and substituting (13) in (11). If a check column is not used, it is desirable to finish one y column and apply the final checks before filling in the other y columns. (The calculations for each y column are independent of those for the others.) If a check column is desired, the procedure and rules given in Section 5 are worded so as to apply to the present case, the y columns being included with the others. There is no restriction on the number of equations or the number of 6'

9. *m EQUATIONS IN n UNKNOWNS. UNUSUAL CASES.* Let there be given m equations in n unknowns. These may be arbitrarily (in any order) labelled 1, 2, . . . , m, and 1,2, . . . , n, respectively, and the above procedure followed as though the equations and unknowns were actually written in the orders indicated by the labels. We shall consider the rows and columns both numbered in the orders in which they are completed in the auxiliary matrix. In general these orders are those in which the equations and unknowns are actually written, so that the labels are superfluous; however, in determining the auxiliary matrix two unusual cases may arise, as follows.[8]

(1.) If in completing a column the diagonal element is found to vanish, the (unlabelled) row chosen to be completed and labelled is one which has a non-vanishing element in this column.

(2.) If in completing a column all of the newly computed elements are found to vanish, that column is left unlabelled; but is (with its zeros) in its final form, and requires no further calculations.

The calculation of the auxiliary matrix is continued until either the rows or the unfinished x columns are exhausted, the total number of labelled rows and columns then being R. (R is the *rank* of the matrix of the coefficients in the given equations.) The m-R unlabelled rows and n-R unlabelled x columns will be called *excess*. Either, both, or neither of the quantities m-R and n-R may vanish, depending upon the given equations.

The one column (or more if there are several y columns as in Section 8) arising from the right hand sides of the equations is now completed, the procedure being identical with that used in completing the last labelled column. If in so doing the computed elements (which lie in excess rows) all vanish, the given equations are compatible; otherwise they are incompatible.

If the equations are found to be compatible, we next omit the excess rows (the equations corresponding to which are superfluous), and determine the final matrix from the remaining rows. Each excess column is treated like the last column, and contributes a column to the final matrix, in which it is labelled -x with the appropriate subscript. We thus obtain R unknowns as linear combinations of the others.

As an example, the given equations:[4]

	X_1	X_2	X_3	X_4	=
	47.126	-74.150	14.222	-64.372	27.304
	-29.312	54.332	-37.998	70.028	-80.476
(14)	32.470	-46.984	- 4.7770	-29.358	-12.934
	- 5.7490	17.257	-30.887	37.842	-66.824
	8.9070	- 9.9090	-11.888	2.8280	-26.586

(OVER)

8 THESE CASES MAY ALSO ARISE IN EVALUATING A DETERMINANT. SHOULD CASE 1 OCCUR, THE SIGN OF THE PRODUCT OF THE DIAGONAL ELEMENTS MUST BE REVERSED IF THE NUMBER OF INVERSIONS OF THE ROW LABELS IS ODD; OTHERWISE, LEFT UNALTERED. SHOULD CASE 2 OCCUR, THE VALUE OF THE DETERMINANT IS ZERO.

have the auxiliary matrix:

(15)

	X_1	X_2	X_3	X_4	=
	47.126	-1.5734	.30179	-1.3660	.57938
	-29.312	8.2113	-3.5502	3.6522	-7.7324
(15)	32.470	4.1056	0	0	0
	- 5.7490	8.2113	0	0	0
	8.9070	4.1056	0	0	0

and the final matrix:

(16)

	$-X_3$	$-X_4$	
X_1 =	-5.2843	4.3805	-11.5871
X_2 =	-3.5502	3.6522	- 7.7324

The first unknown is $x_1 = 5.2843x_3 - 4.3805x_4 - 11.5871$. Since the two labelled rows and columns of the auxiliary matrix are in consecutive order, the labels are omitted, the excess rows and columns being clearly indicated by the block of zeros. Here R = 2.

m final checks can be obtained on each column of the final matrix by substituting the results in the given equations, the other columns in the final matrix and the corresponding columns in the given equations being omitted.

If a check column is desired, the procedure and rules given in Section 5 are worded so as to apply to the present case. In addition it is true that in the auxiliary matrix those elements of the check column which lie in excess rows must vanish if the given equations are compatible.

If the given equations are homogeneous (right hand sides all zero), all columns of the final matrix arise from excess columns of the auxiliary matrix. In those systems ordinarily met in stability and vibration problems m = n = R + 1.

NOTES ON USE OF THE MARCHANT CALCULATOR FOR SOLUTION OF SIMULTANEOUS EQUATIONS BY THE METHOD OF PRESCOTT D. CROUT AS DESCRIBED IN MARCHANT METHOD MM – 434 B 1

Those who frequently use the Crout Method soon memorize the procedure for selecting the proper amounts for calculator entries, in accordance with the precise generalized instructions given in Dr. Crout's paper. However, when the process is to be used infrequently, it is believed that a guide will be useful for showing the manner of progressing with the calculation, as well as for selection of the amounts at each stage of the work.

Such a guide for third and fourth order real matrices is shown on pages 3 and 4 respectively. Except where the amounts are the same in each matrix, the elements of the auxiliary matrix are designated by "row" and "column", with lower case letters whereas those for the given matrix are capitalized.

DECIMALS: For the computed elements to be accurate to as great a number of significant figures as possible, it is advisable to place the Middle Dial Decimal as far to the left as will barely accommodate the largest whole number (at left of its decimal) that exists in the given matrix. The Upper Dial (or Keyboard Dial) Decimal should, likewise, be as far to the left as will barely accommodate the largest whole number (at left of its decimal) in the factor which is to be entered in the Upper Dial (or Keyboard Dial), and the decimal for the Keyboard Dial (or Upper Dial) factor is then set in accordance with the usual rule. There are a few instances, such as when terminal division is by amounts less than 1, in which this rule will need modification.

This plan requires that under most circumstances the Upper and Keyboard Dial Decimals must be re-set for each pair of amounts which are to be multiplied and simultaneously subtracted.

In applying this rule for either the auxiliary or final matrices, it is possible that at times there may not be sufficient capacity to accommodate the accumulation of products of negative expressions which, when subtracted, actually increase the amount of the first entry. In such cases, the eye should be kept upon the dial to be sure that the number that would normally appear in the dial that would be next at the left will not be lost.

Where extreme accuracy is not required, a single decimal setting often can be used which will accommodate all amounts that are likely to be entered in computing the auxiliary (or final) matrix. Such a setting would provide for accommodating the longest whole number likely to be entered in the Upper and Keyboard Dials respectively. The Middle Dial Decimal is then set to conform to these decimals.

CALCULATOR ENTRIES: In the guide shown on pages 3 and 4, the factors are selected from the Matrix in the following order; the left-hand of any pair of factors to be subtractively multiplied is the one in the same row as the element being calculated, and the right-hand of the pair is the one in the same column. It is suggested that this same arrangement be used in calculating; viz, by entering the "row" factor in Keyboard Dial, and entering the "column" factor in Single-Row Keyboard so it appears in the Upper Dial.

In computing the right-hand column of the auxiliary matrix, it is sometimes helpful to imagine that the elements of the corresponding column of the given matrix are multiplied or divided by some power of 10.

(over)

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EVALUATING NEGATIVE AMOUNTS: When the Middle Dial shows an amount in its complementary of "negative" form; i.e., preceded by a succession of 9's, it may be converted to its "positive" form by duplicating the amount in the Keyboard Dial directly below its Middle Dial position, preceding the Keyboard Dial set up with a few 9's. When negatively multiplied by 2, the amount will show in its positive form in the Middle Dial at right of as many ciphers as there are preceding 9's in the Keyboard Dial.

DIVIDING A NEGATIVE NUMBER: In the evaluation of elements at right of principal diagonal, a terminal division is necessary. If the Middle Dial amount prior to such division is in its negative form, a positive quotient is obtained by dividing the negative number as follows:

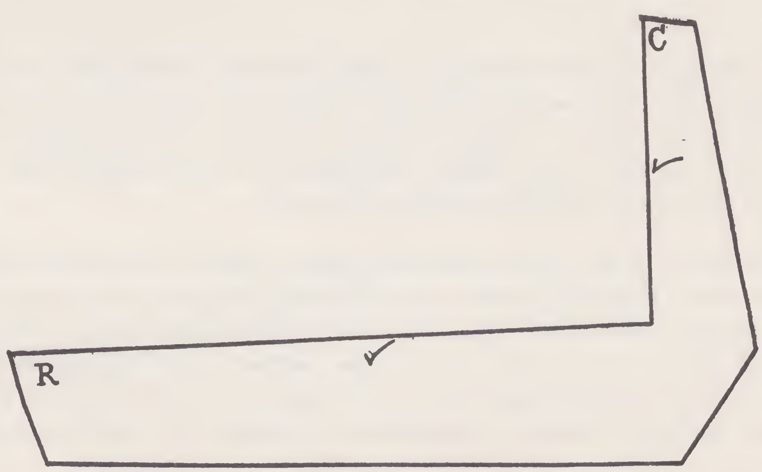
1) Set up the divisor in Keyboard Dial around pre-set decimal and shift carriage until its left-hand figure is directly below the Middle Dial figure that is directly at the right of the succession of 9's.

2) Multiply by any amount that will clear the 9's from left of Middle Dial. (It is not necessary to multiply by the exact amount that will do so. An "over-multiplication" will do no harm.)

3) Without clearing Upper Dial, move Manual Counter Control toward the operator, and divide so that Upper Dial does not clear: Hold down the Stop Key while touching Division Key by a rolling motion of two fingers.

The quotient in positive form will appear in the Upper Dial.

READING ELEMENTS DIRECTLY FROM MATRIX: One of the important benefits of the Crout Method is that intermediate copying to work sheet of all items except those of the Auxiliary Matrix is eliminated. This may be aided by using the device illustrated below, which if made from an easily erasable celluloid material permits pencil check marks to be placed upon the edges so that the particular element being entered is positively identified. In using this device, the vertical edge "C" is placed at right of the column and the horizontal edge "R" is placed directly below the row from which elements are to be read.



(Cont'd.)

THIRD ORDER SIMULTANEOUS EQUATIONS

Guide to Solution by Prescott D. Crout Method

ORIGINAL MATRIX:

X_1	X_2	X_3	S
R1C1	R1C2	R1C3	R1C4
R2C1	R2C2	R2C3	R2C4
R3C1	R3C2	R3C3	R3C4

AUXILIARY MATRIX:

R1C1	r1c2	r1c3	r1c4
R2C1	r2c2	r2c3	r2c4
R3C1	r3c2	r3c3	r3c4

r1c2 = R1C2 ÷ R1C1
r1c3 = R1C3 ÷ R1C1
r1c4 = R1C4 ÷ R1C1
r2c2 = R2C2 − R2C1 • r1c2
*r3c2 = R3C2 − R3C1 • r1c2
**r2c3 = (R2C3 − R2C1 • r1c3) ÷ r2c2
r2c4 = (R2C4 − R2C1 • r1c4) ÷ r2c2
r3c3 = R3C3 − R3C1 • r1c3 − r3c2 • r2c3
r3c4 = (R3C4 − R3C1 • r1c4 − r3c2 • r2c4) ÷ r_3 c_3

The final matrix is:

X_3 = r3c4

X_2 = r2c4 − X_3 • r2c3

X_1 = r1c4 − X_3 • r1c3 − X_2 • r1c2

(*) *If matrix is symmetrical, after recording this element, set up the corresponding diagonal element as a divisor, thus producing the symmetrically opposite element above the diagonal.*

(**) *If matrix is symmetrical, this element is produced by dividing its symmetrically opposite element by the corresponding diagonal element.*

(over)

FOURTH ORDER SIMULTANEOUS EQUATIONS

Guide to solution by Prescott D. Crout Method – Marchant Method MM-434B1

ORIGINAL MATRIX:

X_1	X_2	X_3	X_4	S
R1C1	R1C2	R1C3	R1C4	R1C5
R2C1	R2C2	R2C3	R2C4	R2C5
R3C1	R3C2	R3C3	R3C4	R3C5
R4C1	R4C2	R4C3	R4C4	R4C5

AUXILIARY MATRIX:

R1C1	r1c2	r1c3	r1c4	r1c5
R2C1	r2c2	r2c3	r2c4	r2c5
R3C1	r3c2	r3c3	r3c4	r3c5
R4C1	r4c2	r4c3	r4c4	r4c5

r1c2 = R1C2 ÷ R1C1
r1c3 = R1C3 ÷ R1C1
r1c4 = R1C4 ÷ R1C1
r1c5 = R1C5 ÷ R1C1
r2c2 = R2C2 − R2C1 • r1c2
*r3c2 = R3C2 − R3C1 • r1c2
*r4c2 = R4C2 − R4C1 • r1c2
**r2c3 = (R2C3 − R2C1 • r1c3) ÷ r2c2
**r2c4 = (R2C4 − R2C1 • r1c4) ÷ r2c2
r2c5 = (R2C5 − R2C1 • r1c5) ÷ r2c2
r3c3 = R3C3 − R3C1 • r1c3 − r3c2 • r2c3
*r4c3 = R4C3 − R4C1 • r1c3 − r4c2 • r2c3
**r3c4 = (R3C4 − R3C1 • r1c4 − r3c2 • r2c4) ÷ r3c3
r3c5 = (R3C5 − R3C1 • r1c5 − r3c2 • r2c5) ÷ r3c3
r4c4 = R4C4 − R4C1 • r1c4 − r4c2 • r2c4 − r4c3 • r3c4
r4c5 = (R4C5 − R4C1 • r1c5 − r4c2 • r2c5 − r4c3 • r3c5) ÷ r4c4

The final matrix is:

X_4 = r4c5

X_3 = r3c5 − X_4 • r3c4

X_2 = r2c5 − X_4 • r2c4 − X_3 • r2c3

X_1 = r1c5 − X_4 • r1c4 − X_3 • r1c3 − X_2 • r1c2

(*) *If matrix is symmetrical, after recording this element, set up the corresponding diagonal element as a divisor, thus producing the symmetrically opposite element above the diagonal.*

(**) *If matrix is symmetrical, this element is produced by dividing its symmetrically opposite element by the corresponding diagonal element.*

THE BIRGE-VIETA METHOD
of
FINDING REAL ROOTS OF RATIONAL INTEGRAL FUNCTION

PREFACE: *Few realize tne extent that classical mathematical methods have evolved under the control of the "parameter" (to use a mathematician's word) that pencil-and-paper shall be used in the calculations required by such methods. If the modern calculating machine had been available to the mathematicians of the Renaissance, it is possible that even such a familiar tool as the Briggs Logarithm might not have been developed. Certainly the art would have progressed along far different lines if from the start there had been available a machine that could multiply or divide as rapidly as one could enter amounts in a keyboard.*

The disclosure herein is an interesting example of how an early method, which was discarded because it involved so much numerical computation that was "unfit for a Christian," to quote from a writer of that day, has now been found to possess decided advantages when compared with methods that displaced it. This is because present-day calculating machines remove the drudgery element which caused the method to be relegated to the shelf over 200 years ago.

The method to which we refer was originally proposed by Francis Vieta (1540–1603). Raymond T. Birge, Ph.D., Professor of Physics and Chairman of the Department, University of California, is responsible for re-establishing it as a modern computing tool. Dr. Birge has noted that it possessed many advantages over the methods that have been developed to take its place (merely because of the excessive amount of pencil-and-paper work that it entailed).

In applying the Vieta method to the modern calculating machine, Dr. Birge has reduced it to simple systematic procedure that permits speedy determination of the root under conditions of controlled accuracy.

USES OF THE BIRGE-VIETA METHOD: The method is ideal for finding a real root of the usual algebraic equation when rough approximation of the root is known, particularly if the equation is of higher degree than the second. It is also excellent for solving transcendental equations (those that involve logarithmic or trigonometric functions in combination with analytic functions), particularly when the equations are in such form that substitutions of odd amounts in the equations or in their first derivatives are difficult, Inasmuch as the usual problem of inverse curvilinear interpolation is one of finding the root when the value of the function is a given amount, it will be seen that the Birge-Vieta method is adapted to such work, assuming of course that the tabular values are first expressed as an Interpolation Polynomial of degree "n" that fits n + 1 equidistant values of such tabulated function (See Marchant Method 434-F).

In the case of solving equations involving transcendental functions, tabular values are, likewise, obtained. An Interpolation Polynomial is then fitted to the values and then solved for the desired root. If, however, the equation has a simple first derivative and substitution of amounts in the original equation or its first derivative is not too difficult, the Newton-Raphson Method of obtaining the root is to be preferred.

OUTLINE: It is assumed that the reader is familiar with the usual Horner Synthetic Division process which is described in most College Algebra texts. However, a Note is appended which describes this procedure in a way that will enable it to be understood by a computer who is not familiar with it. (See top of Page 4).

(over)

An algebraic statement of the sample computation is given. This is followed by detailed instructions for performing the work on a Marchant calculator. An Appendix then states the particular advantages of the Birge-Vieta Method, as compared with methods that are ordinarily used for such work.

The symbolism of the Horner Method is employed insofar as possible.

EXAMPLE: Find correctly to nine figures the real root nearest to $x = 1.0$ of the following equation:

$$y = g(x) = x^5 - x - 0.2 \text{ (true value is } 1.044\ 761\ 700_{07})$$

Assume $x = +1 = p_1$ as first approximation of the root.

I Transfer from $g(x)$ to $g'(x - p_1) = g'(x - 1) = g'(u)$
Transfer factor, $p_1 = +1$. Apply Horner Shift for A_0 and A_1
(See Note A. Page 4).

	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	1	0	0	0	−1	−0.2
		1	1	1	1	0
	1	1	1	1	0	$-0.2 = A_0$
		1	2	3	4	
	1	2	3	4	$4 = A_1$	

Therefore $\quad u = -\dfrac{A_0}{A_1} = -\dfrac{-0.2}{4} = +0.05 = x - p_1$

or $\quad x = p_1 - \dfrac{A_0}{A_1} = 1.0 + 0.05 = 1.05 = p_2$, as second approximation.

It will be noted that the above represents the first steps of an ordinary Horner synthetic division. Only A_0 and A_1 need be found.

II Transfer from $g(x) = g''(x - p_2) = g''(x - 1.05) = g''(v)$.

It is a characteristic of this method that the calculations need be carried *only* to the reliability that the ratio of the next coefficients (in this case, B_0 and B_1) is likely to have. A practical rule is to carry *twice* as many *decimal* places in all sums and products used in obtaining B_0 as there are decimal places in the transfer factor. Hence, since 1.05 is the transfer factor, carry B_0 calculations to *four* decimal places. We find, in this problem, three significant figures for B_0, and hence carry all calculations for B_1 to at least *three* significant figures (it is really simpler to carry four and round off to three).

We now return to the *original* coefficients, an essential of the method, and one of its best features from the viewpoint of accuracy control.

$p_2 = 1.05$ transfer factor.

	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	1	0	0	0	− 1	− 0.2
		1.05	1.1025	1.1576_2	$+ 1.2155_{01}$	$+ 0.2262_{76}$
	1	1.05	1.1025	1.1576_2	$+ 0.2155_{01}$	$+ 0.0263 = B_0$
		1.05	2.2050	3.472	4.860	
	1	2.10	3.307	4.629	$5.075 = B_1$	

By inspection $v = -B_0/B_1$ will have two ciphers. Therefore, by rule given, the ratio should be correct to two significant figures.

$$\text{Therefore} \qquad v = -\frac{B_0}{B_1} = -\frac{+0.0263}{5.07} = -\ 0.005187$$

rounded to $-\ 0.0052$

It will be noted that *four* decimal places carried in the B_0 calculations were sufficient to give B_0 to three significant figures, as is desired in order to be sure that B_0/B_1 is correctly calculated to *two* significant figures (i.e., in addition to the two ciphers with which it starts).

$$\text{Therefore} \qquad x = p_2 - \frac{B_0}{B_1} = 1.05 - 0.0052 = 1.0448 = p_3,$$

as second approximation.

III Transfer from $g(x)$ to $g'''(x - p_3) = g'''(x - 1.0448) = g'''(w)$
$p_3 = 1.0448$ transfer factor

As before, since there are four decimal places in $v = -B_0/B_1$ or in p_3, the next transfer factor, we carry eight decimal places in getting C_0, i.e., close to full capacity of a ten-key calculator, so for simplicity the full ten-key capacity is utilized. Then from the C_0 result, carry only six significant figures in computing C_1.

x^5	x^4	x^3	x^2	x^1	x^0
1	0	0	0	-1	-0.2
	1.0448	1.091 607 04	$1.140\ 511\ 035_4$	$+\ 1.191\ 605\ 929_4$	$+\ 0.200\ 189\ 874_6$
1	1.0448	1.091 607 04	$1.140\ 511\ 035_4$	$+\ 0.191\ 605\ 929_4$	$+\ 0.000\ 189\ 875 = C_0$
	1.0448	$2.183\ 21_4$	$3.421\ 53_2$	$+\ 4.766\ 41_9$	
1	2.0896	3.274 82	4.562 04	4.958 02 $= C_1$	

There will be four ciphers in C_0/C_1, therefore carry five or six significant figures.

$$\text{Therefore} \qquad w = -\frac{C_0}{C_1} = -\frac{+0.000\ 189\ 875}{4.958\ 02} = -\ 0.000\ 038\ 296\ 5_4$$

This ratio should be satisfactory to four significant figures. However, we retain five as this is to be the final approximation.

$$\text{Therefore} \quad x = p_3 - \frac{C_0}{C_1} = 1.0448 - 0.000\ 038\ 296_{54} = 1.044\ 761\ 703_{46} = p_4$$

This root should be accurate to nine digits. It is seen that the error is 0.3 in the 9th digit.

A continuation of this process with transfer factor 1.044 761 703 gives $D_0 = +\ 0.000\ 000\ 014_3$ and $D_1 = C_1$(closely enough) $= 4.958$

$$\text{Therefore} \quad -D_0/D_1 = -0.000\ 000\ 002_9 \text{ or } p_5 = p_4 - D_0/D_1 = 1.044\ 761\ 700_1$$

which is correct to ten figures.

An alternate continuation process is to use $p_4 = 1.044\ 761\ 7$ as transfer factor and by double multiplication (see Marchant Method 421A) carry all products to full 20-digit capacity of the calculator, thus producing p_5 correct to 18 or 19 digits (1.044 761 700 075 552 795).

Note that the actual error in p_1 is -0.045, in p_2 is $+0.0052$, in p_3 is $+0.000\ 038$, and in p_4 is $+0.000\ 000\ 003$. Thus, each approximation is correct to about double the number of digits of its predecessor. This is a characteristic feature of the present method. For this reason, p_5 *should* be correct to about 18 digits.

(over)

NOTE A – THE HORNER SHIFT

For those not familiar with the Horner Shift, the procedure is easily understood by reference to the calculation for B_0/B_1 on Page 2, with factors manipulated as below:

Transfer factor p

	x^5	x^4	x^3	x^2	x^1	x^0
Coefficients	a	b	c	d	e	f
of x^n		pa	pm	pn	po	pq
	a	m	n	o	q	$r = B_0$

in which $m = b + pa$, $n = c + pm$, $o = d + pn$, etc. and similarly for the next row that produces B_1.

MATHEMATICAL BASIS OF METHOD

The Birge-Vieta process obtains the value of the function and of its first derivative when the approximate roots (the transfer factors) are substituted for "x". That part of the process which obtains A_0, B_0, C_0, etc., obtains successively more accurate values of the function, and A_0, B_0, and C_0, etc. are these successive values. The step that obtains A_1, B_1, C_1, etc., similarly obtains successively more accurate values of the first derivative when the transfer factors are substituted for "x". This is done, however, not by duplicating the first step with respect to the equation of the first derivative of the function but by taking advantage of partial products and sums developed during the first step. This makes it unnecessary to set up the equation of the derived function.

The successive transfer factors may have the same or different signs. Under some conditions they may alternate in sign.

COMBINING SUBSTITUTION METHODS WITH THE BIRGE-VIETA PROCESS

Inasmuch as A_0, B_0, C_0, etc. are the values of the function when the transfer factor is substituted for "x," and A_1, B_1, C_1, etc., are the first derivatives of the function with respect to "x" when the transfer factor is likewise substituted for "x," there will be cases in which the first two steps of the computation may be more easily done by taking advantage of these facts, using a Table of Powers for direct computation of these amounts. This plan reserves the Birge-Vieta process for cases in which direct substitution is not easy and where the first derivative also is not easy to compute, which by the premise at bottom of Page 1 is its indicated use, anyway. These conditions usually are met when the transfer factor exceeds three figures if a Table of Powers of three-figure amounts is available. It is met with two-digit transfer factors if a Table of Powers is not available, (assuming, of course, that the usual small powers of integers 1 to 9 are known).

A readily available table of "First Ten Powers of the Integers from 1 to 1000" is that of Works Project for Computation of Mathematical Tables, Table MT-1, Information Section, National Bureau of Standards, Washington, D. C.; price 50 cents.

It happens that the example used to illustrate this method is in such form that with the aid of a Table of Powers of three-figure amounts the results of the second section may be obtained somewhat faster by substitution. (The work of the first section is obviously merely a matter of inspection.)

As an example of this straight substitution, let us apply it to this second section. We first note that the powers of 1.05, to four decimals, are $x^5 = 1.2763$ and $x^4 = 1.2155$, (these are the only powers needed for substituting in the equation or in its first derivative).

From this, we have $1.05^5 - 1.05 - 0.2 = 0.0263 = B_0$

and its first derivative $5 \times 1.05^4 - 1 = 5.0775 = B_1$

APPLICATION OF THE BIRGE-VIETA METHOD TO THE MARCHANT CALCULATOR

The skilled computer who prefers to add or subtract mentally, or who wishes to use auxiliary means for such addition or subtraction doubtless would prefer to set up the transfer factor as a constant in the Keyboard Dial and multiply by the various factors as needed. The amounts are then entered on a work sheet exactly as shown in the above analysis.

Others will wish to perform all additions and subtractions on the Marchant. The detailed Marchant operations for this procedure, when applied to the calculation of p_4, are as follows:

OPERATIONS: Decimals: Upper Dial 9, Middle Dial 18, Keyboard Dial 9. Use any 10 column Marchant.

Inasmuch as the coefficient of x^5 is 1.0, the calculator computation is started for development of the x^3 column; thus,

(1) Enter 1.0448 in Keyboard Dial and multiply by transfer factor (1.0448).
Copy 1.091 607 04 from Middle Dial to x^3 column.

(2) As there is no amount to add to this, the normal adding step is skipped. Shift to Position 10, clear Upper and Keyboard Dials, and copy Middle Dial amount (1.091 607 04) into Keyboard Dial, clear Middle Dial and multiply by transfer factor (1.0448).
Copy $1.140\ 511\ 035_4$ from Middle Dial to x^2 column.

(3) Repeat Step (2) with Keyboard Dial entry of 1.140 511 035.

(4) Clear Keyboard Dial, shift to Position 10, set up 1.0, and subtract.
Copy $0.191\ 605\ 929_4$ from Middle Dial to x^1 column.

(5) Repeat Step (2) with Keyboard Dial entry of 0.191 605 929.

(6) Clear Keyboard Dial, shift to Position 10, set up 0.2, and subtract.
Copy C_0 (0.000 189 875) to x^0 column.

(7) Clear all dials, enter 1.0448, and multiply by 2.0.
Copy 2.0896 from Middle Dial to x^4 column.

(8) Shift to Position 10, clear Keyboard Dial and copy Middle Dial amount (2.0896) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(9) Shift to Position 10, clear Keyboard Dial, enter 1.091 607 04, and add.
Copy 3.274 82 from Middle Dial to x^3 column.

(10) Copy Middle Dial amount (3.274 82) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(11) Shift to Position 10, clear Keyboard Dial, enter 1.140 511 035, and add.

(12) Copy Middle Dial amount (4.562 04) to Keyboard Dial, clear Upper and Middle Dials, and multiply by transfer factor (1.0448).

(13) Shift to Position 10, clear Keyboard Dial, enter 0.191 605 929, and add.
Copy 4.958 03 from Middle Dial as C_1.

(over)

(14) Clear dials, enter C_0 (0.000 189 875) and add.

(15) Enter C_1 (4.958 03) and divide.
$-w = 0.000\ 038\ 296$ appears in Upper Dial.

(16) Clear Middle and Keyboard Dials, shift to Position 10, enter 1.0448, and add.

(17) Enter 1.0 and negatively multiply by Upper Dial amount that is at right of decimal (.000 038 296), reducing it to ciphers.
Root (1.044 761 704) appears in Middle Dial.

That the error is "4" in the 10th significant figure, whereas the analysis on Page 3 shows it to be "3," comes about because the Marchant does not drop off right-hand figures in producing $4.766\ 41_9$ of Step (12). Slight variations of this sort from the analysis are to be expected. The root, however, is still accurate to 9 figures, which is all that this stage of the computation is expected to obtain.

The continuation of the process with transfer factor 1.044 761 704, if it is desired to go so far, may be done in the same manner as above.

Reference is made in the analysis to "double multiplication" with carrying all products to 20 digits. This is assisted by the means mentioned in Marchant Method 421 A, "Multiplication of Large Factors."

APPENDIX -- ADVANTAGES OF THE BIRGE-VIETA METHOD

Dr. Birge gives the following reasons why the Vieta process, when adapted to a calculator, is to be preferred, as compared with the more commonly used Ruffini-Horner Method. These advantages are in addition to the extra speed of the Vieta process because of there being fewer steps.

(1) One always deals with the *same* original coefficients (which often contain relatively few significant figures), instead of with constantly new sets of coefficients, which inevitably get more complex, as in the R-H Method.

(2) Any *error* in the calculation affects *only* the particular transfer being made, and can never affect the final result. The same thing is true for the Newton iteration method, and constitutes the greatest advantage of that method. Thus, due to an error, a certain approximation may be poorer than the preceding approximation, but this fact immediately shows up in the *next* approximation. In other words, $p_1\ p_2\ p_3$ should constitute a series of numbers that rapidly settles down to a *constant* value, just as $x_1\ x_2\ x_3$ etc. in Newton's iteration method (for square roots, etc.) rapidly become constant.

But in the R-H method, *any* error makes the new function incorrect, and since we *then* proceed to get the root 0 the *new* function, the final result is necessarily incorrect. In other words, any such error carries through to the end. This advantage of the Vieta method over the R-H method can scarcely be overemphasized, and should be alone sufficient to make the R-H method completely obsolete.

(3) In the Vieta method the transfer factors $p_1\ p_2$ etc., are all approximately the same size, and since the original coefficients are always used (advantage 1), *all* corresponding products and sums appearing in successive Horner shifts are approximately the same. Hence we do not need to figure the position of the decimal point, after the first Horner shift has been made. This fact is of great advantage in avoiding errors, and it results in much time saved.

(4) As already stated, one needs to calculate *only* the first two coefficients of each new function, whereas all coefficients must be calculated in the R-H method.

(5) In calculating these first two coefficients, we do not need, at first, to get the various sums and products to the *final* desired accuracy (as *is* necessary in the R-H method).

APPROXIMATING POLYNOMIAL FROM DIFFERENCE ARRAY (STIRLING METHOD)

REMARKS: It is often desired to obtain an algebraic expression for a function that is determined by the relation that a series of tabulated amounts bears to corresponding values of the independent variable. When values of the latter are taken at equidistant points so that an array of differences may be set up, an equation in the form of an Approximating Polynomial may be readily obtained. If the nth difference of the array is constant, the Approximating Polynomial will represent the function correctly provided differences up to, and including those of the nth order are taken into account. If there are differences in the array which are of higher order than those taken into account, the Approximating Polynomial will approximate the function insofar as it can be done by a polynomial of degree "n".

Obtaining an Approximating Polynomial by means outlined herein provides the most rapid method of fitting an equation to non-periodic tabulated data of scientific and statistical computations. It is assumed that the data are "smoothed"; that is to say, the obvious errors of observation are eliminated as is the case when the tabulated values are taken from a curve or determined by least-squares methods. If functions appear in periodic form, the Approximating Polynomial found by the method herein is generally suitable only for showing one quarter-period (approximately) of the periodic function. Fourier Series analysis is generally employed for obtaining equations of periodic functions.

OUTLINE: The Approximating Polynomial described herein has the form

$$(1) \quad y = a_0 + a_1u + a_2u^2 + a_3u^3 \ \ldots\ldots\ + a_nu^n$$

in which the "a" values are coefficients to be determined, and "u" represents the independent variable reduced to the initial condition that u= 0 when $y = a_0$ and that the difference between tabulated values of the independent variable, in terms of "u", is "1". For example, in the table showing the relation between x and y, below, the values of "u" are shown in the middle column assuming that the 0 point of "u" is to be at x = 0.3. It is obvious that if an Approximating Polynomial in the form of (1) is obtained, it is easy to convert it to one that shows y as a function of x. This simple transformation is not discussed herein.

EXAMPLE:

FUNCTION			DIFFERENCES				
x	u	y	1st	2nd	3rd	4th	5th
0.	−3	1.00000					
			−4865				
0.1	−2	0.95135		1547			
			−3318		−299		
0.2	−1	0.91817		1248		100	
			−2070 ($d'_{-\frac{1}{2}}$)		−199 ($d'''_{-\frac{1}{2}}$)		−32 ($d^{V}_{-\frac{1}{2}}$)
0.3	0	0.89747		1049 (d''_0)		68 (d''''_0)	
			−1021 ($d'_{\frac{1}{2}}$)		−131 ($d'''_{\frac{1}{2}}$)		−32 ($d^{V}_{\frac{1}{2}}$)
0.4	+1	0.88726		918		36	
			− 103		− 95		
0.5	+2	0.88623		823			
			720				
0.6	+3	0.89343					

(over)

The method exemplified herein will be applied to the tabulated values listed on the previous page which are shown with differences. An Approximating Polynomial is to be obtained in such form that it will show optimum accuracy in the vicinity of x = 0.3. This value is then chosen as the base point for obtaining the "a" coefficients, so "u" is set at 0 when x = 0.3.

The formula used is that of Stirling and is chosen because it is the easiest to apply. The Bessel formula* gives somewhat more accurate results in the region that is half-way between the equidistant tabular values. This difference, however is exceedingly slight so that rarely will it be advisable to go to this refinement. The Newton formula* is useful for obtaining an Approximating Polynomial when only the values at the top of a table are obtainable. However, even in this case the Stirling Method may be used if it is satisfactory to extrapolate probable differences upward from the known differences.

In the Approximating Polynomial (1), Page 1, the Stirling formulas for coefficients, up to consideration of 5th differences, are below:

(2) $a_1 = \frac{1}{2}\,(d'_{-\frac{1}{2}} + d'_{\frac{1}{2}}) - 1/12\,(d'''_{-\frac{1}{2}} + d'''_{\frac{1}{2}}) + 1/60\,(d^{V}_{-\frac{1}{2}} + d^{V}_{\frac{1}{2}})$

(3) $a_2 = \frac{1}{2}\,d''_0 - 1/24\,d''''_0$

(4) $a_3 = 1/12\,(d'''_{-\frac{1}{2}} + d'''_{\frac{1}{2}}) - 1/48\,(d^{V}_{-\frac{1}{2}} + d^{V}_{\frac{1}{2}})$

(5) $a_4 = 1/24\,d''''_0$

(6) $a_5 = 1/240\,(d^{V}_{-\frac{1}{2}} + d^{V}_{\frac{1}{2}})$

The terms up to 4th differences appear in Scarborough: Numerical Mathematical Analysis, 1930 edition, Page 80. Those for 5th differences were supplied by courtesy of Dr. Raymond T. Birge, Professor of of Physics, University of California, to whom we are also indebted for other helpful data in connection with this process.

The nomenclature of equations (2) to (6), inclusive, applies to the preceding difference array and is further explained in Marchant Method 439 E2. It will be noted that certain factors are repeated or bear simple ratios to others.

For ordinary computing, any terms that do not affect the final result in one place at the right of the one that is to be retained would be omitted. If values of the polynomial are desired close to the centering point, it is often possible to shorten the work if advantage is taken of this principle. In this case, it is not possible to do this if 5-place accuracy is desired without uncertainty within the range u = -1 to u = +1, because the maximum effect of the 5th difference is noted in coefficient a_3 as 0.000 013 3 (see below) so it would affect 6th place by 13.

If accuracy to the number of places of the tabulated values is desired up to the limits of the values from which differences are taken; i.e., the extreme range of the tabulated values, the higher-order coefficients must be taken to more places than those of lower order. For example, at the extreme range of the table, u = ±3. As the coefficient a_5 multiplies u^5, or 243, it is evident that a_5 must be carried to a sufficient number of places so that the error of its right-hand digit when multiplied by 243 will not affect 6th place.

(*) Scarborough: Numerical Mathematical Analysis, 1930 edition, the Johns Hopkins press, pages 80 – 81, gives coefficients for Newton, Bessel, and Stirling formulas up to and including 4th differences. Values including those due to 5th differences will be supplied upon application.

Using these principles, the coefficients are obtained as follows:

$a_1 = -0.015\ 455 - (-0.000\ 275) + (-0.000\ 011) = -0.015\ 191$

$a_2 = +0.005\ 245 - 0.000\ 028\ 3 = +0.005\ 216\ 7$

$a_3 = -0.000\ 275 - (-0.000\ 013\ 33) = -0.000\ 261\ 67$

$a_4 = +0.000\ 028\ 33$

$a_5 = -0.000\ 002\ 667$

The Approximating Polynomial, accordingly, is

$$(7)\ y = 0.89747 - 0.015191u + 0.0052167u^2 - 0.00026167u^3 + 0.000028333u^4 - 0.000002667u^5$$

To show how closely this approximates the tabulated function, when u varies from −3 to +3, its values are computed to six places.

x	u	y Computed to 6 places	Tabulated y	6th place error
0	−3	1.000 000	1.000 00	0
0.1	−2	0.951 351	.951 35	1
0.2	−1	0.918 170	.918 17	0
0.3	0			
0.4	+1	0.887 260	.887 26	0
0.5	+2	0.886 229	.886 23	1
0.6	+3	0.893 429	.893 43	1

MARCHANT CALCULATOR APPLICATION

No exemplification of the details of Marchant application to this work is given because it embodies the simplest of calculator manipulation. Because work of this sort is usually infrequently done and because some of the factors of equations (2) to (6) inclusive are repetitions or bear simple numerical ratios to others, it is usually advisable to evaluate each factor individually, copying the amounts to work sheet and summing them afterwards. For these reasons, the accumulation of partial products is not recommended, though this procedure should undoubtedly be followed if there is a great volume of the work to be done.

In nearly all cases, except where it is desired to obtain an empirical formula (see below), it is usually satisfactory to use the function in terms of "u"; thus, for direct interpolation and related work the "x" is converted to the corresponding "u" before applying the formula, and in cases of inverse interpolation and the like, the "x" is obtained after the "u" has been found.

APPROXIMATION OF FUNCTIONS BY POLYNOMIALS

Polynomials of the type considered herein for the representation of a tabulated function have not been given the consideration in mathematical literature that their importance warrants. It is believed that this is due to the usual comparatively laborious process of setting them up by solving systems of linear equations, which has long been the conventional method of converting n tabulated values into a power series of degree n. Now that it is recognized that they are much more easily obtained from their difference arrays, more and more uses are certain to be found for them.

One principal use of these polynomials is to provide means of handling complicated analytical or transcendental functions in which substitution is difficult owing to the complexity of the terms. Equidistant values are established, sufficient to determine the Approximation Polynomial. A few intermediate values are obtained for a later check of the error. The Polynomial is then used in

(over)

place of the function for which it is a substitution. When given the "u" value, the "y" is obtainable by direct substitution in the polynomial. When given the "y" value, the "u" is easily obtained by the Birge-Vieta Method (see Marchant Method 434 D)

The above-described procedure is particularly helpful in cases where the differential or integral values of a complicated function are desired. Many of these cannot be integrated directly and differentiation is often difficult. If the expression is approximated as a polynomial, however, it is a simple matter to obtain successive differential or integral forms and without the discountinuities which use of the original expression might entail. A characteristic of the Approximation Polynomial is that its graph has minimum curvature.

The use of these polynomials in cases of large volume of interpolation, such as in table preparation, is obvious, though in such instances the procedure of Marchant Method 439 E1 should be compared. Inverse interpolation is easily handled by using the Birge-Vieta Method for solving for "y". (Compare also Marchant Methods 439 J2, 439 H, and 439 J1.

The polynomials readily lend themselves to extrapolation provided it is understood that the uncertainty of the result increases (sometimes rapidly) as one leaves the region contained between the extreme values from which the differences are tabulated.This effect becomes increasingly serious as the degree of the polynomial increases.

The polynomials also provide a way of exploring the effect of the powers of the independent variable in cases of experimental tabulated data, thus leading directly to an empirical formula to express the relationships. Obviously, if the coefficient of, say, the third power of x (not of u) is large and those of other powers are negligible, the experimenter will be on the lookout for influences that vary according to the cube of the independent variable. Care must be taken not to accept too literally the significance of the polynomial as a working formula, however, because an empirical formula should, if possible, have some physical meaning or reasonable basis for being in the form used.

If the polynomial shows comparatively large coefficients of x (not of u) for certain powers, an empirical formula, however, may generally be set up using those coefficients and powers only. The values of y corresponding to the tabulated x's may then be computed from this new polynomial and compared with the original tabulated values. The residuals then may be considered as n values of another new polynomial containing only the powers that are to be retained in the proposed empirical formula. By solving these as a system of linear equations, applying least squares methods, a modification is obtained of the coefficients of the powers that are to be retained in the empirical formula. This modified formula then becomes the improved empirical formula.*

The above-described procedure is the usual one of taking advantage of an approximating polynomial (power series) as a base for an empirical formula. Another case in which such a polynomial may be converted into a simplified empirical formula is that in which the successive coefficients follow a definite law, indicating a convergent series, which represents some other function such as an exponential, trigonometric, etc.

(*) Steinmetz: Engineering Mathematics, 3rd Edition, McGraw-Hill Publ. Co., Pages 215-16. See also Marchant Method 434B2 and Marchant Method 434B1 (Page 7, Section 6). These relate to the Crout Method for solving such systems of equations.

NOGRADY METHOD FOR SOLUTION OF CUBIC EQUATIONS

REMARKS: *The application of the Birge-Vieta Method (See MM-434D) to the solution of a cubic (third degree) equation gives the real root that is nearest to the first approximation. The work must then be repeated for other real roots. No imaginary roots are found. Special study has been given by Henry A. Nogrady* to the problem of obtaining all roots of such equations, both real and imaginary. Complete exposition of the method is given in his monograph, "A New Method for the Solution of Cubic Equations."* By aid of a table included in this book, the work is greatly simplified.*

The description herein exemplifies the use of the Marchant calculator when applied to the general cubic equation having three real roots, or having one real root and two conjugate complex roots. Modification to fit cases of two real roots, one real and two non-conjugate complex roots, and three complex roots, as well as tests for recognizing in what classification any equation comes, is fully covered in the Nogrady monograph, which is assumed to be in possession of the reader.

OUTLINE: The general cubic equation

(1) $ax^3 + bx^2 + cx + d = 0$

where a, b, c, and d are any numbers, is transformed into

(2) $y^3 + py + q = 0$

by substituting

(3) $$\frac{3ac - b^2}{3a^2} = p \text{ and } \frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a} = q$$

(2) then becomes

(4) $z^3 + nz + n = 0$

by substituting

(5) $$\frac{p^3}{q^2} = n$$

If n is real, eq. *(4)* has at least one real root. Its value is tabulated in the Nogrady monograph as z_1. By substitutions not outlined herein, the other roots of *(4)* are

(6) $$z_2 = \frac{z_1}{2}\left(-1 + \sqrt{\frac{z_1 - 3}{z_1 + 1}}\right)$$

(7) and $$z_3 = \frac{z_1}{2}\left(-1 - \sqrt{\frac{z_1 - 3}{z_1 + 1}}\right)$$

(*) *"A New Method for the Solution of Cubic Equations" by Henry A. Nogrady, ~~29 - 18 Taylor Ave.,~~ Detroit, Michigan. For sale by the author, price $1.25 postpaid.*
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(over)

When z_1, z_2, and z_3 are found, the corresponding y's are found by multiplying the z's by the ratio q/p.

The corresponding x's are found by subtracting b/3a from the y's.

The computation is expedited if the following terms are evaluated in the order named:

$3a$, $3ac$, $3a^2$, $27a^3$, p, bc, q, $b/3a$, q^2, n, q/p, $(z_1 - 3)/(z_1 + 1)$. Extract root of previous amount, and then evaluate the y's and x's. This listing of elements of the computation does not comprise the bettering of the table value of z_1 (see Eq. 6).

EXAMPLE I

Find roots to 5 places of $x^3 + 2x^2 + 10x - 3 = 0$

By substitutions outlined above

$$y^3 + 8.66667y - 9.07407 = 0$$

and

$$z^3 + 7.90592z + 7.90592 = 0$$

From Nogrady Table, Page XXIV, the nearest $n = 7.911462$ for which the corresponding root z_1 is -0.906.

This value is improved to six figures by the following process:

(8) Six-figure value of $z_1 = \dfrac{2z_1^3 - n}{3z_1^2 + n}$ *NOTE: A four-figure value requires only linear interpolation except at certain extremes of table.*

in which $z_1 = -0.906$ and $n = 7.90592$; or Six-figure $z_1 = -0.905950$

from which $z_2 = .45298 + 2.91919\ i$ and $z_3 = .45298 - 2.91919\ i$

Multiplying these z's by q/p, we have

$$y_1 = 0.94854;\ y_2 = -0.47427 + 3.05642\ i;\ y_3 = -0.47427 - 3.05642\ i$$

Subtracting b/3a, we have

$$x_1 = 0.28187;\ x_2 = 1.14094 + 3.05642\ i;\ x_3 = 1.14094 - 3.05642\ i$$

The latter two roots, because of symmetry, are termed Conjugate Complex Roots. The symbol "i" indicates $\sqrt{-1}$.

OPERATIONS: Decimals; Upper Dial 6, Middle Dial 11, Keyboard Dial 5. Use any Marchant 8 or 10 column model.

NOTE: Because the coefficients are simple integers, certain operations listed below normally would be omitted. For sake of completeness, however, they are listed. Whether a multiplication or division is positive or negative depends upon the sign of the factors and whether their product is to be added or subtracted. The procedure given below requires this obvious modification in the case of examples that have different signs from the equation considered herein.

(1) Set up in Keyboard Dial "a" (1.00000) and multiply by 3.
Copy "3a" (3.00000) from Middle Dial to Work Sheet.

(2) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "c" (10.00000).
Copy "3ac" (30.00000) from Middle Dial to Work Sheet.

(3) Clear Upper and Middle Dials, and multiply by "a" (1.00000).
Copy "$3a^2$" (3.00000) from Middle Dial to Work Sheet.

(4) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "a" (1.00000).

(5) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by 9.
Copy "$27a^3$" (27.00000) from Middle Dial to Work Sheet.

(6) Clear all dials, set up in Keyboard Dial "3ac" (30.00000), shift to 7th position, and depress Add Bar. Then depress Subtract Bar, set up "b" (2.00000) in Keyboard Dial, and reverse multiply by "b" (2.00000).

(7) Change Keyboard Dial to read "$3a^2$" (3.00000), and divide.
Copy "p" (8.66667) from Upper Dial to Work Sheet.

(8) Clear all dials, set up in Keyboard Dial "b" (2.00000), and multiply by "c" (10.00000).
Copy "bc" (20.00000) from Middle Dial to Work Sheet.

(9) Clear Upper and Middle Dials and multiply by "b" (2.00000).

(10) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "2b" (4.00000).

"$2b^3$" (16.00000) appears in Middle Dial, but it need not be copied to Work Sheet.

(11) Change Keyboard Dial to read "$27a^3$" (27.00000) and divide.

(12) Clear Keyboard and Middle Dials, set up in Keyboard Dial "bc" (20.00000), shift to 7th position, depress Add Bar and then depress Subtract Bar, change Keyboard Dial to read "$3a^2$" (3.00000), move Manual Counter Control toward the operator, and depress Division Key in the manner that will not cause Upper Dial to clear.

(13) Clear Keyboard and Middle Dials, set up in Keyboard Dial "d" (3.00000), shift to 7th position, depress Add Bar, and then depress Subtract Bar, change Keyboard Dial to read "a" (1.00000), and inasmuch as "d" is negative the Manual Counter Control will be left as it was in Step 12; i.e., toward the operator. Depress Division Key in the manner that will not cause Upper Dial to clear. Move Manual Counter Control away from operator.

NOTE: It will now be observed that Upper Dial shows a negative amount. This is evaluated as a positive amount and copied to Work Sheet as "q" (-9.07407).

(14) Clear all dials, set up "b" (2.00000), and with carriage in 7th position, depress Add Bar.

(over)

(15) Change Keyboard Dial to read "3a" (3.00000) and divide.

Copy "b/3a" (0.66667) from Upper Dial to Work Sheet.

(16) Clear all dials, set up "q" (9.07407) and multiply by "q" (9.07407).

Copy "q^2" (82.33875) from Middle Dial to Work Sheet.

(17) Clear all dials, set up "p" (8.66667) and multiply by "p" (8.66667).

(18) Clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial, clear Middle Dial, and multiply by "p" (8.66667).

(19) Change Keyboard Dial to read "q^2" (82.33875) and divide.
Copy "n" (7.90592) from Upper Dial to Work Sheet.

(20) From Table of Nogrady Roots, Page XXIV, the nearest "n" is 7.911462 for which corresponding root "z_1" is -0.906.

NOTE: The computation for improving this root to 0.905950 by formula 8 is obvious. It is taken to 5 places as -0.90595.

(21) Clear all dials, set up in Keyboard Dial "q" (9.07407) and, with carriage in 7th position, depress Add Bar. Change Keyboard Dial to read "0" (8.66667) and divide.

Copy "q/p" (1.04701) from Upper Dial to Work Sheet.

(22) Clear all dials, set up in Keyboard Dial "z_1-3" (3.90595) and with carriage in 7th position, depress Add Bar. Change Keyboard Dial to read "$z_1 + 1$" (0.09405) and divide.

Copy $(z_1-3)/(z_1+1)$ or (-41.53057) from Upper Dial to Work Sheet.

(23) Extract Square Root of -41.53057 by a Marchant Table, producing a five-figure root of 6.4444 which is expressed as 6.4444 i, indicating that it is the square root of a negative number.

NOTE: This square root may be improved, if desired, to 6.44442 i.

(24) Clear all dials, set up in Keyboard Dial "$z_1/2$" (0.45298) and multiply by square root from Step 23 (6.44442).

Copy coefficient of i (2.91919) from Middle Dial to Work Sheet, thus completing all figures from z_2 and z_3.

(25) Clear all dials, set up in Keyboard Dial "q/p" (1.04701) and multiply by z_1 (0.90595) and the real and imaginary parts of z_2 and z_3 (0.45298) and (2.91919) producing

y_1 (0.94854); y_2 (-0.47427 + 3.05642 i);
and y_3 (-0.47427 - 3.05642 i).

(26) Clear all dials. With carriage in 7th position, set up y_1 (0.94854), and add. Set up "b/3a" (0.66667) and, with Non-Shift Key down, reverse multiply by 1.

x_1 (0.28187) appears in Middle Dial.

(27) Clear Middle Dial and touch Add Bar. Set up the real part of y_2 and y_3 (0.47427) and add, thus completing values for

x_2 (1.14094 + 3.05642 i)
x_3 (1.14094 - 3.05642 i)

EXAMPLE II

Find roots to 5 places of $x^3 - 7x + 6 = 0$

This is in the form of $y^3 + py + q = 0$, so the operations following Step No. 15 need only be done with certain obvious deletions. The outline is below:

$n = p^3/q^2 = -343/36 = -9.52778.$

From Table, nearest "n" is -9.516913 for which z_1 is -1.169.

This value is improved by (8) to

$$z_1 = \frac{2\ (-1.169^3) + 9.52778}{3\ (-1.169^2) - 9.52778} = \frac{6.33276}{-5.42810} = -1.16667$$

$$q/p = 6/-7 = -0.85714$$

$$\sqrt{(z_1 - 3)/(z_1 + 1)} = \sqrt{25} = 5$$

$$z_2 = -0.58333 \cdot 4 = -2.33333$$

$$z_3 = -0.58333 \cdot -6 = 3.50000$$

$$x_1 = y_1 = -0.85714 \cdot -1.16667 = 1.$$

$$x_2 = y_2 = -0.85714 \cdot -2.33333 = 2.$$

$$x_3 = y_3 = -0.85714 \cdot 3.50000 = -3$$

The Marchant operations are similar to most of those following Step 16 of Example I.

STARTING VALUES FOR MILNE-METHOD INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER, OR OF SECOND ORDER WHEN FIRST DERIVATIVES ARE ABSENT

THE METHOD OF TAYLOR'S SERIES

(Compare also Marchant Method MM-437E1

REMARKS: *Marchant Method MM-437E3 describes the Milne Method of integrating differential equations in the form of $dy/dx \equiv y' = u(x,y)$, and Marchant Method MM-437E4 similarly relates to equations in the form $d^2y/dx^2 \equiv y'' = v(x,y)$. Each requires that a few starting values at equidistant values of x be known. Integration then proceeds from these starting values by quadrature processes.*

This method relates to obtaining starting values for use in MM-437E3 (and 437E4) when the functions y' (or y") may be differentiated with respect to x, and each derivative so found may likewise be differentiated until a sufficient number of derivatives of successively higher order are obtained for substitution in a Taylor's Series that will give the desired y's to the accuracy required. In cases where this successive differentiation is impracticable, see Marchant Method MM-437E1.

It is assumed that a knowledge of Taylor's Series is had. The method herein is subject to the limitations in the use of this series that have to do with convergence, continuity of the successive derivatives with respect to both x and y, etc. However, the use of this series for obtaining the few terms necessary to provide starting values for subsequent Milne-Method integration introduces certain matters which are not discussed in usual texts. Such subjects are covered herein (see particularly Note 2).

OUTLINE: The Taylor's Series in the form (1) below

$$(1)\quad f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f''''(x_0)}{4!}(x-x_0)^4 + R$$

is written in the equivalent form (2), as follows:

$$(2)\quad y = y_0 + y'_0h + \frac{y''_0}{2}h^2 + \frac{y'''_0}{3!}h^3 + \frac{y''''_0}{4!}h^4 + R \text{ (Remainder)}$$

in which $h = (x-x_0)$, $0 < h < 1$

Inasmuch as the initial condition gives the value of $y = y_0$ when $x = x_0$ in functions of the type $dy/dx = u(x,y)$, and furthermore gives $y' = y'_0$ in the case of functions of the type $d^2y/dx^2 = v(x,y)$, and the successive differentiations give the y''_0, y'''_0, y''''_0 .... etc., everything is known for substituting in (2), thus obtaining y corresponding to the value of x which differs from x_0 by the amount h, except for the Remainder.

If h is chosen sufficiently small and enough terms are taken, the Remainder may be reduced to as small an amount as we please, which provides accuracy control of the method.

Four consecutive starting values of y which correspond to equally-spaced values of x are required to start the Milne-Method solution by the Milne 3-term formulas. Six values are required if the 5-term formulas are used, etc. As one of the required values is the initially known y, three and five new values have to be computed, respectively.

(over)

To aid in the identification of the y's found by Taylor's Series with the starting values of the Milne-Method integrations of MM-437E3 & 437E4, the table below is given. It also shows how h in (2) is obtained. As to this, the Taylor's Series procedure, as outlined herein, uses a variable interval h to obtain the successive y's. The intervals, however, are in multiples of s, which is equal to the "h" of the Milne-Method integrations; that is to say, certain intervals of the Taylor's Series integrations (those which are x_0-s and x_0+s, respectively) are the same as the h of the Milne-Method integrations, but one is twice as large when obtaining a Taylor's Series set of starting values for Milne-Method 3 or 5-term formulas, and another is three times as large for obtaining the last starting value for the Milne 5-term formulas. This will be clear from inspection of the following table and the example.

Taylor's Series Notation			Milne-Method Notation of MM-437E3 and 437E4
x	y	Interval h	
Corresponding to 3-term Milne formulas:			
x_{-1}	y_{-1}	$x_0 - s$	y_{-3}
x_0	y_0	x_0 initial values	y_{-2}
x_1	y_1	$x_0 + s$	y_{-1}
x_2	y_2	$x_0 + 2s$	y_0
Corresponding to 5-term Milne formulas:			
x_{-2}	y_{-2}	$x_0 - 2s$	y_{-5}
x_{-1}	y_{-1}	$x_0 - s$	y_{-4}
x_0	y_0	x_0 initial values	y_{-3}
x_1	y_1	$x_0 + s$	y_{-2}
x_2	y_2	$x_0 + 2s$	y_{-1}
x_3	y_3	$x_0 + 3s$	y_0

The foregoing procedure is recommended for obtaining the few starting values for subsequent integration by the Milne (or other) method. This is mentioned because some texts when describing the Taylor's Series procedure suggest proceeding from each x to the next; that is to say, each new y with its corresponding x is taken as a new pair of initial values. Such a plan has the disadvantage that a new set of derivatives must be evaluated for each pair of x,y values. Furthermore, it cannot take full advantage of the simplification of work when obtaining such derivatives that occurs when the initial x is zero. In such a case, terms of each derivative which contain x as a factor drop out and the series reduces to that of Maclaurin.

The example given herein shows the use of the Taylor's Series method for obtaining starting values for integrating by the Milne 3-term formulas of MM-437E3 $dy/dx = -xy$. Similar procedure would be followed if a double integration of $d^2y/dx^2 = (x^2 - 1)y$ of MM-437E4 were to be made. In the latter case, y'_0 is also known as one of the initial conditions so everything is required for substitution in (2), just as is the case when using the method for the first-order equation of MM-437E3.

EXAMPLE: Find starting values for integrating by Milne 3-term formulas $dy/dx \equiv u(x,y) = -xy$ with initial condition that $y = 1$ when $x = 0$, with intervals in multiples of $s = 0.1$, corresponding to Milne-Method integration when $h = 0.1$; the final y_2 is to be accurate to 5 places; i.e., 6th place error is to be less than 5.

We start by obtaining the successive derivatives of $y' \equiv dy/dx = -xy$. Because of the fact that the initial $x = x_0$ is zero, it is unnecessary to find the derivatives in terms of x and y. Time is saved by expressing each succeeding derivative in terms of the former, as follows:

$y' = -xy$	as $x_0 = 0$ and $y_0 = 1$		$y'_0 = 0$
$y'' = -xy' - y$	" " " " " " " "		$y''_0 = -1$
$y''' = -xy'' - 2y'$	" " " " "	y'_0 " 0	$y'''_0 = 0$
$y'''' = -xy''' - 3y''$	" " " " "	$y''_0 = -1$	$y''''_0 = 3$
$y^{v} = -xy'''' - 4y'''$	" " " " "	$y'''_0 = 0$	$y^{v}_0 = 0$

and similarly $y^{vi} = -15$, $y^{vii} = 0$, and $y^{viii} = 105$.

Substituting in (2) the various values of the derivatives at $x = x_0$, as well as the initial y_0, it will be seen that terms having odd powers of h vanish, leaving

(3) $y = 1 - h^2/2 + 3h^4/4! - 15h^6/6! + 105h^8/8! + \text{Remainder}$

As this is an alternating converging series (see Note 1), the Remainder is less than the first term discarded. Thus, if we discard the term containing h^6, the error will have an upper bound, as follows:

When obtaining	y_{-1}	R will be less than	$-15 \times .000001/720 = -.00000002$
" "	y_1	" " " " "	$-15 \times .000001/720 = -.00000002$
" "	y_2	" " " " "	$-15 \times .000064/720 = -.00000133$

Inasmuch as this is less than the permitted error of 0.000005 and it is obvious that fewer terms will not be satisfactory, the desired values of y_{-1} to y_2 are found using terms of (3) up to and including the one containing h^4, as follows:

	h	1st term	2nd term	3rd term	Total
y_{-1}	-0.1	1.00000	$-(-0.1)^2/2 = -0.05$	$3(-0.1)^4/24 = 0.0000125$	0.99501
y_0	0	1.00000	--	--	1.00000
y_1	0.1	1.00000	$-(0.1)^2/2 = -0.05$	$3(0.1)^4/24 = 0.0000125$	0.99501
y_2	0.2	1.00000	$-(0.2)^2/2 = -0.2$	$3(0.2)^4/24 = 0.0002$	0.98020

Inasmuch as this integration may be done analytically, the prediction of the value of the Remainder may be verified. The correct y_2 to 8 places is 0.98019867. The Remainder, R, in (3) is thus, -0.00000133, which is the same as the upper bound that was predicted. However, if more places are taken, the absolute value of R will be found to be minutely less.

NOTE 1: Though knowledge of the principles of Taylor's Series is assumed, it may be helpful to mention that when the series has its terms of the same sign, as distinct from the alternating series of the example, a strictly mathematical computation of the Remainder ordinarily is not easy.

Though an explicit expression for it is sometimes possible, the usual means of expressing the Remainder in (2) is in the form of the first rejected term of the series, with the exception that the derivative is taken not at $x = x_0$, but at some value of x, unknown, which lies between x_0 and $x_0 + h$. To compute the maximum possible value of the Remainder under this condition requires that the derivative of required order be known in terms of x and y (or the y for any x approximated) and also that such value of x (with its corresponding approximated y) be used as will cause the Remainder to be a maximum. Though this provides a true upper bound, it may be much higher than the actual Remainder, so more terms of (2) would be taken than actually would be required if the true Remainder were known.

(over)

Now, when obtaining starting values for subsequent continuance by the Milne Method, it is not important to know the exact value of the Remainder in (2); it is only necessary to know that it does not exceed a certain permissable error in the final y.

The most practicable way of determining this is to compute a few extra terms, plot their values with respect to the order of the term, and, by inspection, obtain some idea of the ratio that the value of each succeeding term bears to its predecessor. By adding the values and projecting them to a limit -- they will usually converge rapidly -- it may be readily determined whether their sum will cause the total error to be less than the permissable error.

NOTE 2: It will have been noted that the starting values for the Milne integration are obtained herein by the expedient of first integrating backwards by the interval $h = -s$ when values for the Milne 3-term formulas are being found, and by the additional backward integration by the interval $h = -2s$ in the case of the 5-term formulas. This differs from customary practice in Taylor's Series integration of always integrating ahead from $x = x_0$. It thus provides an unwanted value (or values) and fails to compute, instead, a value (or values) that can be used.

This objection does not hold if the computer establishes the initial values because in such a case these values would merely be taken greater by s (or 2s) than the starting value that was required to be known.

Regardless of this, the reason for recommending the within procedure; i.e., integrating backwards at the start, is that doing so reduces the coefficients of the last retained term so that desired accuracy usually may be had by using fewer terms of the series, with the further advantage that much differentiating is avoided. The time so saved more than offsets the slight extra time of computing an additional Milne-Method value (or two), for the latter is done very rapidly once the Milne-Method computation is started.

To show the advantage of starting the Taylor's Series integration by backward integration instead of forward, there is given below the error of y_3 if it were obtained by forward integration by Taylor's Series in the conventional manner.

When obtaining	y_1	R will be less than	(as previously computed)	-0.00000002
" "	y_2	" " " "	" "	-0.00000133
" "	y_3	" " " "	$-15 \times .000729/720$	-0.00001519

which is 3 times that permitted. As a consequence, in this case, the term containing h^6 in (3) would have to be retained if the conventional plan of integrating forward were followed.

The effect described becomes increasingly significant when obtaining starting values for Milne-Method integration by the 5-term formulas, or larger. In the case of the 5-term formulas, for example, the remainders when computing the necessary y_4 and y_5 are often so great as substantially to invalidate the process. Even in the case of a series which converges as rapidly as (3) and discarding the term containing h^8, it will be found that if the plan outlined herein is followed (using backward integration to obtain y_{-2} and y_{-1} and thence forward successively to y_3), 6-place accuracy of the last value is had, whereas if forward integration from y_0 to y_5, inclusive, is used according to the conventional method, only 3-place accuracy of the final value is obtained.

CURVILINEAR INTERPOLATION BY LAGRANGEAN COEFFICIENTS

Example, with Table, Supplied by and Reproduced by
courtesy of
George Rutledge, Ph.D., Professor of Mathematics
Massachusetts Institute of Technology

REMARKS: *In the preparation of scientific and mathematical tables it is customary to make observations or calculations only sufficiently close together to clearly show the trend of the function. Interpolation for in-between points may then be done by "curvilinear interpolation" so that the points found fall on a smooth curve connecting the known values. These interpolated values are then included in the table. After issuance of the table, the ordinary user employs "straight line" interpolation for intermediate values (see Marchant Methods 439A1, 439B, 439C, 439A2, 439A3), though when great accuracy is desired the "curvilinear" method will be used.*

This method relates to "curvilinear interpolation" when given five equidistant points. Similar procedures enable interpolation to a closer degree when given 7 points, or up to 17, if desired. Further information is contained in the paper of Dr. Rutledge (with Prescott Crout), "Tables and Methods for Extending Tables for Interpolation Without Differences," Jour. of Math. and Phys., Vol. IX, No. 3, 1930.

Whereas the method herein refers to curvilinear interpolation without use of "differences" (see Marchant Method 419), the conclusion must not be drawn that interpolation by using differences is any less exact. Differencing has an advantage that it indicates errors in the tabulation from which the interpolation is made. Direct and Inverse Curvilinear Interpolation, using differences, is fully discussed in the works of Dr. L.H. Comrie, late Supt., H.M. Nautical Almanac Office and now Managing Director, Scientific Computing Service, Ltd., London. (List of references supplied upon application.)

EXAMPLE: Let us assume that in preparation of a table of 7 place logarithms, calculations are made as follows:

For "n" equals 4.8	$\log_{10}{}^{n}$	equals	0.68124	12
4.9	"	"	.69019	61
5.0	"	"	.69897	00
5.1	"	"	.70757	02
5.2	"	"	.71600	33
5.3	"	"	.72427	59

It is desired to interpolate for $\log_{10}{}^{n}$ for "n" equals 5.04 (as one step in the preparation of a more detailed table).

NOTE: As all values are known, the solution to a known result clearly exemplifies the method.

OUTLINE: By reference to the table of Rutledge Crout 5 point Lagrangean coefficients attached hereto, it will be observed that interpolation may be made forward from 5.0, with the interval (lambda) being 0.4, or it may be made backward from 5.1 with the interval being −0.6. The double computation is essential because it is only through agreement in the two results that any conclusion as to accuracy is justified.

(over)

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The result desired is the algebraic sum of the products of the five logarithms and their respective "coefficients" as taken from the table for intervals 0.4 and –0.6 respectively. The point from which interpolation takes place is the median.

OPERATIONS: Decimals: Upper Dial 7; Middle Dial 14; Keyboard Dial 7. Set Tab Key 7.

INTERPOLATING FORWARD FROM 5.0

In this case the median is 5.0 and the 5 points are 4.8, 4.9, 5.0, 5.1 and 5.2.

(1) Set up in Keyboard Dial value of function for 4.8 (.68124 12) and multiply by coefficient K_{-2} for interval .4 (.0224000).

(2) Clear Upper and Keyboard Dials, set up in Keyboard Dial the value of function for 4.9 (.69019 61) and negatively multiply by coefficient K_{-1} (–.15360 00).

(3) Proceed as above for the following multiplications:

K_0 .69897 00 × .80640 00
K_1 .70757 02 × .35840 00
K_2 .71600 33 ×–.03360 00 (multiply negatively)

Middle Dial shows $\text{Log}_{10}5.04$ equals 0.70243 05 plus.

(It may be noted that "straight-line" interpolation gives the erroneous value .70241 01)

INTERPOLATING BACKWARD FROM 5.1

In this case the median is 5.1, and the points are 4.9, 5.0, 5.1, 5.2 and 5.3.

As the interpolation is backward, the coefficients for lambda equal to –.6 as indicated by the bottom line of the table are used.

(1) Set up in Keyboard Dial value of function for 4.9 (.69019 61) for interval –.6 and negatively multiply by coefficient K_{-2} (–.04160 00).

(2) Clear Upper and Keyboard Dials, set up in Keyboard Dial the value of function for 5.0 (.69897 00) and multiply by coefficient K_{-1} (.58240 00).

(3) Proceed as above for the following multiplications:

K_0 .70757 02 × .58240 00
K_1 .71600 33 × –.14560 00 (multiply negatively)
K_2 .72427 59 × .02240 00

Middle Dial shows $\text{Log}_{10}5.04$ equals 0.70243 06 minus.

Agreement may be expected (on the basis of error formulas) in this range to 7 places with an uncertainty of 1 in the final place if 7 place data is used.* With 10 place data we may expect (on the basis of the same error formulas) agreement in the forward and backward interpolation to essentially 9 places. These error predictions vary according to the range or degree of curve between the five points.

NOTE: Using a similar 7 point interpolation method and data from 10 place British Association Tables of logarithms, the value for 5.04 is found by this method to be 0.70243 05364 plus, with agreement to 10 places in forward and backward interpolating.

(*) The Rutledge-Crout error formulas show that this error, when seven place data are used in this part of the table, cannot exceed .00000 007.

(Cont'd)

RUTLEDGE CROUT
FIVE POINT LAGRANGEAN COEFFICIENTS
(Exact Values)

Interval λ	K_{-2}	K_{-1}	K_0	K_1	K_2	Interval* λ
0.1	.00783 75	-.05985 00	.98752 50	.07315 00	-.00866 25	-0.1
.2	.01440 00	-.10560 00	.95040 00	.15840 00	-.01760 00	- .2
.3	.01933 75	-.13685 00	.88952 50	.25415 00	-.02616 25	- .3
.4	.02240 00	-.15360 00	.80640 00	.35840 00	-.03360 00	- .4
.5	.02343 75	-.15625 00	.70312 50	.46875 00	-.03906 25	- .5
.6	.02240 00	-.14560 00	.58240 00	.58240 00	-.04160 00	- .6
.7	.01933 75	-.12285 00	.44752 50	.69615 00	-.04016 25	- .7
.8	.01440 00	-.08960 00	.30240 00	.80640 00	-.03360 00	- .8
.9	.00783 75	-.04785 00	.15152 50	.90915 00	-.02066 25	- .9
λ	K_2	K_1	K_0	K_{-1}	K_{-2}	λ

*The values K_{-2}, K_{-1}, K_0, K_1, K_2 for negative values of λ are indicated by the bottom line of this table.

Thus for λ equals –.6, K_{-2} equals –.0416000, etc.

DIRECT INTERPOLATION AND SUB–TABULATION
(IF FOURTH DIFFERENCES DO NOT EXCEED 1000)

EXPLANATORY APPENDIX TO MARCHANT-METHOD MM-439E1

A NOTE ON OBTAINING 4TH DIFFERENCES FOR USE WITH "COMRIE THROW-BACK" IN EXAMPLE IV

Reference was made in the second paragraph of the "Remarks" section, Page 1, of Marchant Method MM-439E1, to the fact that in sub-tabulation it is not necessary to obtain third and fourth differences, except at infrequent intervals, and then only in order to obtain their general range as a guide to the selection of method or as a means of obtaining the 4th difference correction of Example IV.

Inasmuch as a 4th difference must be known before the "4th difference correction" can be determined, it might appear that the statement is inconsistent, because obviously 4th differences will normally vary somewhat from interval to interval.

Actually, however, in ordinary computing practice, it will be found that the 4th difference correction generally can be obtained without the necessity of completely tabulating the 3rd and 4th differences. This is because the large majority of functions which are tabulated to the number of places used in ordinary computing, – rarely more than 7 places – will have no great variation in 4th differences; that is to say, a small 5th difference.

By following the procedure below, the tabulation of 4th differences for every interval may usually be avoided.

A plan that does this is to obtain 4th differences at about every fifth or tenth interval and observe their trend, plotting them graphically and obtaining the 4th differences for the intermediate intervals from the curve so drawn.

This will ordinarily give quite as accurate a 4th difference as would actual differencing at each interval, because the graphical method eliminates the error forced into the 4th difference due to rounding up of the right-hand digit of the tabulated function. Such round-ups can affect the right-hand digit of the 4th difference by as much as 8.

In considering the precision of this approximation, it is to be noted that in the computation of the interpolates only "$0.184 \times$ 4th differences" is used. This is additional justification for the procedure of eliminating 4th differencing for every interval when the work falls within the class of Example IV.

INVERSE CURVILINEAR INTERPOLATION BY "DIVIDED DIFFERENCES"

REMARKS: *When it is desired to determine the argument (independent variable) that corresponds to some function (dependent variable) which lies between values for which there are corresponding tabulated arguments, recourse must be had to Inverse Interpolation. If it is desired to take into account the fact that the function lies on a smooth curve connecting its tabulated values, the process is designated Inverse Curvilinear Interpolation.*

The method herein, basically due to LaGrange, is suitable for infrequent calculations. If extensive work is to be done, such as when transferring tables from one set of coordinates to another, the Comrie Two-Calculator Method is much more suitable. The method herein is also useful for obtaining a real root of a function which is known only by its tabular values or which is difficult to determine from its analytic form.

The method is LIMITED to cases in which the first differences of the function do not change in sign during the intervals from which data are obtained to compute the intermediate functions used in the process. From this it follows that the method is applicable only to cases in which the tabulated values are either CONTINUALLY INCREASING or CONTINUALLY DECREASING within the intervals from which data are obtained to compute the intermediate functions.

If the tabulated values are those of an analytic function of usual form having continuous derivatives up to but not including the order of differences which tend to disappear, then the argument so obtained will correspond to the specified value of the function.

EXAMPLE: Find X when Y = 0.32999 in the table below. (For other purposes to be described later, the table is shown with its differences, as computed by Marchant Method MM-419).

Argument X	Function Y	Differences 1st (d')	2nd (d'')	3rd (d''')	4th (d'''')
1.1	0.45360		-454		3
		-9124		92	
1.2	0.36236		-362		3
		-9486		95	
1.3	0.26750		-267		1
		-9753		96	
1.4	0.16997		-171		2
		-9924		98	
1.5	0.07073		- 73		1
		-9997			
1.6	-0.02924				

By inspection of differences, it will be observed that they are continuous and that the first differences (d') do not change sign.

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OUTLINE OF CALCULATION

The array of differences on the preceding page shows that 5th differences are substantially zero. For precise work, the 4th differences should be taken into account. For reasons outlined on Page 4, they should not be disregarded as would be the case with direct interpolation. The ordinates to be considered in this computation, therefore, are those contained between the left ends of the two diagonal lines; i.e., from 1.2 to 1.6 inclusive. It will be noted that the column of 4th differences tends to vary, probably because of forcing errors from rounding of the functions. For purposes of a correction after completion of the work, the 4th differences are plotted and a "true amount" of 1.3 at argument 1.4 is taken.

RETABULATION WITH DIVIDED DIFFERENCES: The argument and function columns are interchanged below, though to prevent confusion of nomenclature they are left with their original designations. Certain "divided difference" functions f_1, f_2, etc., are included in this array, the values of which are obtained in the next step of this outline. The "y" column has also been altered to express the difference between the actual "y's" and the one for which the unknown "x" is to be found; i.e., 0.36236 − 0.32999 = 0.03237, 0.26750 − 0.32999 = −0.06249, etc. The argument that is to be found is now a function of the value of the new "y" when it is zero; i.e., f (0).

y	x = f(y)	f_1	f_2	f_3	f_4
		"Divided Differences" inserted from next steps			
.03237 (y_0)	1.2 (fy_0)				
		−1.054185 $(f_1)_{\frac{1}{2}}$			
−.06249 (y_1)	1.3 (fy_1)		−0.150006 $(f_2)_1$		
		−1.025326 $(f_1)_{1\frac{1}{2}}$		−0.206491 $(f_3)_{1\frac{1}{2}}$	
−.16002 (y_2)	1.4 (fy_2)		−0.089787 $(f_2)_2$		0.072495 $(f_4)_2$
		−1.007658 $(f_1)_{2\frac{1}{2}}$		−0.178102 $(f_3)_{2\frac{1}{2}}$	
−.25926 (y_3)	1.5 (fy_3)		−0.036937 $(f_2)_3$		
		−1.000300 $(f_1)_{3\frac{1}{2}}$			
−.35923 (y_4)	1.6 (fy_4)				

The computed figures were actually made to 8 places, though they are rounded to 6 in the above array. This practice is followed throughout this method and serves to explain why the six-place figures, when substituted, may not produce the exact six-place result that is shown.

CALCULATION OF DIVIDED DIFFERENCES: The values of f_1, f_2, etc., of the above array are computed in accordance with the plan of the diagram below, in which the actual "f" values shown are individually described and computed on the next page.

"y"	"x"	"Divided Difference" Functions			
		f_1	f_2	f_3	f_4
y_0	$f(y_0)$				
		$f(y_0y_1)$			
y_1	$f(y_1)$		$f(y_0y_1y_2)$		
		$f(y_1y_2)$		$f(y_0y_1y_2y_3)$	
y_2	$f(y_2)$		$f(y_1y_2y_3)$		$f(y_0y_1y_2y_3y_4)$
		$f(y_2y_3)$		$f(y_1y_2y_3y_4)$	
y_3	$f(y_3)$		$f(y_2y_3y_4)$		
		$f(y_3y_4)$			
y_4	$f(y_4)$				

The actual computations in accordance with the outline at bottom of preceding page are as follows:

$$(1)\quad f(y_0y_1) = \frac{f(y_0) - f(y_1)}{y_0 - y_1} = \frac{-0.1}{0.09486} = -1.054185$$

$$(2)\quad f(y_1y_2) = \frac{f(y_1) - f(y_2)}{y_1 - y_2} = \frac{-0.1}{0.09753} = -1.025326$$

$$(3)\quad f(y_2y_3) = \frac{f(y_2) - f(y_3)}{y_2 - y_3} = \frac{-0.1}{0.09924} = -1.007658$$

$$(4)\quad f(y_3y_4) = \frac{f(y_3) - f(y_4)}{y_3 - y_4} = \frac{-0.1}{0.09997} = -1.000300$$

$$(5)\quad f(y_0y_1y_2) = \frac{f(y_0y_1) - f(y_1y_2)}{y_0 - y_2} = \frac{-0.028860}{0.19239} = -0.150006$$

$$(6)\quad f(y_1y_2y_3) = \frac{f(y_1y_2) - f(y_2y_3)}{y_1 - y_3} = \frac{-0.017667}{0.19677} = -0.089787$$

$$(7)\quad f(y_2y_3y_4) = \frac{f(y_2y_3) - f(y_3y_4)}{y_2 - y_4} = \frac{-0.007358}{0.19921} = -0.036937$$

$$(8)\quad f(y_0y_1y_2y_3) = \frac{f(y_0y_1y_2) - f(y_1y_2y_3)}{y_0 - y_3} = \frac{0.060219}{0.29163} = -0.206491$$

$$(9)\quad f(y_1y_2y_3y_4) = \frac{f(y_1y_2y_3) - f(y_2y_3y_4)}{y_1 - y_4} = \frac{-0.052850}{0.29674} = -0.178102$$

$$(10)\quad f(y_0y_1y_2y_3y_4) = \frac{f(y_0y_1y_2y_3) - f(y_1y_2y_3y_4)}{y_0 - y_4} = \frac{0.017333}{0.39160} = -0.072495$$

CALCULATION OF ARGUMENT: The formula applying is $f(0) = f(y_0) - y_0(f_1)_{\frac{1}{2}} + y_0y_1(f_2)_1 - y_0y_1y_2(f_3)_{1\frac{1}{2}} + y_0y_1y_2y_3(f_4)_2 \cdots\cdots$

Substituting values, we have

(1) fy_0		1.2
(2) $-y_0(f_1)_{\frac{1}{2}}$		+ .034124
(3) $+y_0y_1(f_2)_1$		+ .000303
(4) $-y_0y_1y_2(f_3)_{1\frac{1}{2}}$		+ .000067
(5) $+y_0y_1y_2y_3(f_4)_2$		+ .000002
		1.23449643 (to 8 places)
	4th difference correction *(see below)*	.00000077
		1.23449566

NOTE:
Take "y" values from left-hand column of middle array on Page 2.

Take "f" values from same array at right.

FOURTH DIFFERENCE CORRECTION: As stated at top of Page 2, the 4th difference at argument 1.4 is taken as 1.3 instead of as 2. Inasmuch as the 5th item, above, shows the effect of the 4th difference, this reduction of 35% should reduce item (5) correspondingly, and as it is positive, the interpolate is decreased to 1.23449566. For reasons given at bottom of Page 4, it is not good practice, (provided the function values are rounded, as is usually the case) to take this value to more figures than 1.23450, with an uncertainty of 1 or 2 in the last figure.

(over)

NOTES

NUMBER OF FIGURES TO BE TAKEN: The value obtained is the most precise result that this method makes possible. It contains no uncertainty because of a Remainder Term, which is substantially non-existant. However, it cannot be used to the number of places shown because the function values contain an error from rounding off their final figures. Because of this, the rule is to take the number of places of the change in argument to no more than the number of significant figures in the first difference of function, and if the latter begins with 1, 2, or 3, to take one less place. For example, in this case there are four significant figures in the first difference of the function (Y), which begin with a number greater than 1, 2, or 3. The number of places in the change in argument cannot be taken as more than 4. (The change in argument is the set of figures at right of 1.2; viz., 3449566). These can be taken only to four figures, making the value that can be used as 1.23450 with an uncertainty of 1 or 2 in the last figure.

WHY METHOD LIMITED TO CASES WHERE FIRST DIFFERENCE OF FUNCTION DOES NOT CHANGE SIGN: If it changes sign, the function rises and falls in the interpolation interval, thereby having a tangent point in which a change in Y produces a change in X at an infinitely greater rate. Should the intervals be small, the differences that are divisors in producing f_1's (see middle of Page 2), may produce extraordinarily large f_1's, leading to asymptotic conditions of unreliability. Not all cases where first differences change sign will produce these effects, but it is necessary to stress this as a limitation to the application of the method.

SELECTION OF ARGUMENTS TO BE TAKEN INTO ACCOUNT: A frequently heard criticism of the Divided Difference Method is that it provides no means of knowing how many function values should be taken into account. This point is covered herein by means of the difference array and the diagonal lines mentioned at top of Page 2. It might be thought that the higher order differences which involve small amounts could be disregarded by applying the same rules as for direct interpolation; viz., disregard 2nd, if they are less than 4, 3rd, if less than 8, 4th, if less than 12 (for Descending Difference formulas). This does not apply in the case of the inverse operation unless the corresponding derivative of function with respect to argument is "1" or more. As it is not easy to determine the higher-order derivatives from differences when the variables are interchanged (though mathematical methods render this comparatively simple) it is believed advisable to include in the calculation all orders of difference that appear in the array. This makes it unnecessary to compute a Remainder Term of the interpolation polynomial because the next higher order of difference would be non-existant (for the usual type of functions found in computing practice).

Quite a satisfactory result is obtained in the example used if 4th differences are disregarded. A skilled computer would know that this would be true because he would make a rough estimate of the probable fourth derivative at argument 1.4 by considering it as of about the order of magnitude of the slope of the tangent to the curve of "3rd differences divided by cube of interval." However, application of this method is somewhat uncertain and inasmuch as very little extra labor is required to include all significant differences, the method herein has been developed in that manner.

CHECK OF VALUE FOUND BY DIRECT INTERPOLATION: If by MM-439E1 or otherwise, we interpolate a function between tabular values that corresponds to argument of 1.234496, it will be found to be 0.33000 ±, the exact amount depending upon the formula that is used. However, use of argument 1.23450 produces an interpolate of 0.32999+. It is not possible to say that one is more "correct" than the other, as it is not unusual for the values of the inverse and direct interpolations to vary slightly in an opposite manner.

MARCHANT COMPUTATION

The previous outline appears somewhat formidable, but the actual calculation on the Marchant is simple, requiring surprisingly few steps, as follows:

OPERATIONS: Decimals; Upper Dial 8, Middle Dial 16, Keyboard Dial 8, No. 9 Tab Key depressed. Use any 10 column Marchant. (The method is also adapted to 8 column Marchants.)

(1) **Compute** the differences in the array at bottom of Page 1, using Marchant Method MM-419, and at the same time compute the differences between any two of the arguments whose differences are not shown on the array; i.e., $y_0 - y_2 = 0.19239$; $y_1 - y_3 = 0.19677$; $y_2 - y_4 = 0.19921$; $y_0 - y_3 = 0.29163$; $y_1 - y_4 = .29674$; $y_0 - y_4 = 0.39160$.

(2) **With carriage in 9th position,** set up in Keyboard Dial the constant interval of the argument (0.1) and add. Similarly, set up the first divisor $y_0 - y_1$ (0.09486) and divide.

$(f_1)_{\frac{1}{2}}$ (1.05418512) appears in Upper Dial, which is copied to array at middle of Page 2, with minus sign, as it is a quotient of a minus amount divided by a plus amount.

(3) **Similarly,** obtain all values of f_1, f_2, f_3, and f_4, the factors being given on Page 3. Clear all dials.

(4) **Obtain** the amounts $y_0y_1, \ldots\ y_0y_1y_2y_3$ as shown in the formula for computing argument (see bottom of Page 3) as follows: Set up in Keyboard Dial y_0 (0.03237) and multiply by y_1 (0.06249), copying answer y_0y_1 from Middle Dial (-0.00202280); clear Upper and Keyboard Dials, transfer Middle Dial amount to Keyboard Dial; clear Middle Dial and multiply by y_2 (.16002), copying answer $y_0y_1y_2$ (0.00032369) and in the same manner obtain $y_0y_1y_2y_3$ (-0.00008392) and $y_0y_1y_2y_3y_4$ as (0.00003015), affixing signs according as the individual factors are plus or minus.

(5) **Clear dials, shift to 9th position, set up in Keyboard Dial** $f(y_0)$ (1.2) and add.

(6) **Clear Upper Dial,** set up $(f_1)_{\frac{1}{2}}$ (1.05418512) and multiply by y_0 (0.03237) because the first factor is minus, the second is plus, and the product is to be subtracted (see Item 2 of formula at bottom of Page 3), so the product is to be added.

(7) **Clear Upper and Keyboard Dials,** and similarly multiply $(f_2)_1$ (0.15000561) by y_0y_1, (0.00202280), obtaining the latter value from Step 4, above.

(8) **Clear Upper and Keyboard Dials,** and similarly multiply $(f_3)_{1\frac{1}{2}}$ (0.20649059) by $y_0y_1y_2$ (0.00032369), obtaining the latter value from Step 4, above.

(9) **ClearUpper and Keyboard Dials,** and similarly multiply $(f_4)_2$ (0.07249451) by $y_0y_1y_2y_3$ (0.00003015).

Inverse Interpolate for 0.32999 (1.23449643) appears in Middle Dial. After 4th difference correction is made (see Page 3), value is 1.23449566.

NOTE: The number of figures of the above that may be taken as correct, assuming rounding off of functional values, is discussed on Page 4.

Whether or not the multiplications of Steps 6, 7, 8, or 9 are made so as to add the products, as formed, or to subtract them, depends upon the sign of the product in accordance with the signs of its factors, and also whether or not the product is to be added to or subtracted from the previous amount. It is assumed that the computer is familiar with this matter.

THE A. C. AITKEN METHOD OF CURVILINEAR INTERPOLATION WITH EQUAL OR UNEQUAL INTERVALS OF THE ARGUMENT

REMARKS: Straight-line interpolation between two points may be performed as described in Marchant Method MM-439A1 according to the following well-known principle: If given two functional values u_a and u_b corresponding to arguments "a" and "b", an intermediate value u_x corresponding to argument "x" between "a" and "b", may readily be obtained by a continuous calculator operation as follows:

$$u_x = \frac{u_a(b - x) - u_b(a - x)}{b - a}$$

This relation also holds if "x" does not lie between "a" and "b"; that is to say, the formula is suitable as a means of straight-line extrapolation.

A. C. Aitken* has published a method of curvilinear interpolation based upon the above relationship. The method first obtains a set of approximations to u_x by applying the above formula successively for pairs of values ab, ac, ad, ae, etc., and then by considering the approximations thus found to be a new set of functional values, the process is again applied to the new set to give a second set of approximations still closer to the desired value, and so on until successive iterations produce no change in the interpolate to the number of places it is desired to retain.

The method is equally suitable for interpolation if the functional values are tabulated at equal or unequal intervals of the argument; i.e., the method is suitable for either direct or inverse interpolation.

The particular field of use for the method appears to be cases in which (1) interpolation must be made to an argument-value bearing fractional relationship to the tabulated arguments of more than three decimals, because there are readily available tables of Bessell, Everett, and LaGrange coefficients** for fractional values differing by .001, or (2) for interpolation when values are tabulated at unequal intervals of the argument, in which case the method serves the same purpose as the Method of Dividend Differences (See Marchant Method MM-439J2). Aitken has shown that his process is the same as using the general divided-difference formula to the same order of differences as the number of stages of his process.

Particularly do we recommend that this method be given consideration in cases of interpolation at unequal intervals of the argument. The exceedingly simple manner in which the example of Marchant Method MM-439J2 responds to the Aitken method is shown herein on page 4.

(*) *A. C. Aitken, Edinburgh Math. Soc. Proc., s. 2, v. 3, 1932, p. 56. An excellent synopsis of the method written by J. R. Womersley appears in Math. Tables and Other Aids to Computation, Vol. II, No. 15, p. 112, July 1946, National Research Council, Washington, D. C.*

(**) *of Marchant Methods MM-439G, 439E1, 439H; 439J1, 228, which relate to various methods of direct and inverse curvilinear interpolation. MM-228 gives 7-place Lagrangean 5-point interpolation coefficients for values of p from 0 to 2, by .001. U. S. Bureau of Standards, Mathematical Tables, Project, offers more extensive tables of Lagrangean coefficients, and many of its tables of other functions show supplementary tables of Bessell and Everett coefficients. L. J. Comrie's tables and explanatory information, "Interpolation and Allied Tables," H. M. Stationery Office, York House, Kings Way, London, W.C. 2, are indispensable to anyone working in this field.*

(over)

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CASE 1: INTERPOLATION WITH EQUAL INTERVALS OF THE ARGUMENT

EXAMPLE: The example used is the one given in Aitken's paper. Given the tabulated values "u" below, to find the value of "u" for argument 0.68327. Development of the computation is also shown.

Argument	u	Stage (1)	Stage (2)	Stage (3)	Stage (4)	Parts
-2	5772 1566					-2.68327
-1	5608 8546	53 339732.4				-1.68327
0	5447 8931	. . 371133.4	39 2588.8			-0.68327
1	5289 2109	. . 401987.0	. . 2128.1	227 4.0		0.31673
2	5132 7488	. . 432306.6	. . 1674.9	... 6.6	. . 3.2	1.31673
3	4978 4499	. . 462107.2	. . 1229.9	... 9.3	. . 3.2	2.31673

DECIMALS: Upper Dial 8 and 1, Middle Dial 16 and 9, Keyboard Dial 8 and 1. Use 10-Column Marchant. Division-Clear Lever toward operator.

OPERATIONS: In the following, multiply with reference to the Upper Dial decimal at 8 and Keyboard Dial decimal at 1. Enter divisors at Keyboard Dial decimal 8 and read quotients at Upper Dial decimal 1. Use "parts" as multipliers.

1) Compute the column of "parts" by setting up the argument to which interpolation is to be made (.68327) and subtracting it to produce its complement (..99.31673) in the Middle Dial. Then enter the greatest positive argument (3.0000) in Keyboard Dial and add, producing 3 – .68327 = 2.31673 in Middle Dial. The other values in this case are obtained by inspection, though if there are unequal intervals of the argument, after each addition, the amount added should then be subtracted to restore the negative constant (..99.31673) in the Middle Dial.

2) Each value for Stage (1) is computed as a continuous cross-multiplication and division process as follows:

$$(5772\,1566 \times -1.68327) - (5608\,8546 \times -2.68327) = 5333\,9732.4$$

This step has no division operation as the argument interval is 1.

The successive computations are thus

$$\frac{(5772\,1566 \times -0.68327) - (5447\,8931 \times -2.68327)}{2} = 5337\,1133.4$$

$$\frac{(5772\,1566 \times 0.31673) - (5289\,2109 \times -2.68327)}{3} = 5340\,1987.0$$

and so on to

$$\frac{(5772\,1566 \times 2.31673) - (4978\,4499 \times -2.68327)}{5} = 5346\,2107.2$$

(continued)

The well-known rule of signs governs the Marchant entries; that is to say, in the preceding expression, the second multiplier (-2.68327) is not entered in negatively as would be indicated by its minus sign. Instead, it is entered directly as a positive amount because the product (4978 4499 × -2.68327) is to be subtracted. This is equivalent to adding (4978 4499 × 2.68327) to the amount that is already in the Middle Dial.

3) Values for Stage (2) are similarly computed as follows:

$$(33\,9732.4 \times -0.68327) - (37\,1133.4 \times -1.68327) = 39\,2588.8$$

$$\frac{(33\,9732.4 \times 0.31673) - (40\,1987.0 \times -1.68327)}{2} = 39\,2128.1$$

and so on to

$$\frac{(33\,9732.4 \times 2.31673) - (46\,2107.2 \times -1.68327)}{4} = 39\,1229.9$$

4) The values for Stage (3) are similarly computed as follows:

$$(2588.8 \times 0.31673) - (2128.1 \times -0.68327) = 2274.0$$

and so on to

$$\frac{(2588.8 \times 2.31673) - (1229.9 \times -0.68327)}{3} = 2279.3$$

5) The values for Stage (4) are as follows:

$$(4.0 \times 1.31673) - (6.6 \times 0.31673) = 3.2$$

$$\frac{(4.0 \times 2.31673) - (9.3 \times 0.31673)}{2} = 3.2$$

which completes the iteration to 5339 2273.2

NOTES: In Step (5) obvious reduction of length of multipliers would be made because only two significant figures are desired.

It is not necessary to write the individual steps in formula form as outlined above because the computer soon has the relationships in mind so it is only necessary to make the computation and copy the respective values as shown under the heading, "EXAMPLE."

(over)

CASE 2: INTERPOLATION WITH UNEQUAL INTERVALS OF THE ARGUMENT

If the arguments are tabulated at unequal intervals, the method is applied in a similar way.

EXAMPLE: The example of Marchant Method MM-439J2, solved therein by divided differences, is easily and quickly solved as below for argument 96.94:

Argument	u	Stage (1)	Stage (2)	Parts
96.70	1.449 8542			−.24
96.90	1.456 0764	1.457 3208		−.04
97.02	1.459 8272	.... .340	...252	.08
97.25	1.467 0530	.... .591	...252	.31

The value of "u" corresponding to 96.94 is thus seen to be 1.4573252.

The process is obvious, but we show a few typical computations, as follows:

From Stage (1)

$$\frac{(1.449\,8542 \times -.04) - (1.456\,0764 \times -.24)}{96.90 - 96.70} = 1.457\,3208$$

$$\frac{(1.449\,8542 \times .08) - (1.459\,8272 \times -.24)}{97.02 - 96.70} = 1.457\,3340$$

From Stage (2)

$$\frac{(208 \times .08) - (340 \times -.04)}{97.02 - 96.90} = 252$$

NOTES: If the final value is inserted in the column of amounts of any stage in line with the argument (untabulated) to which it corresponds, it will be found that the amounts in that stage converge to the value. This is a helpful broad check on the progress of the work.

As in all interpolation, it is recommended that the points used be those which centrally surround the unknown value to the extent that this is possible. In cases where it cannot be done, as at the end of a table, the convergence is slower and rounding errors magnify themselves. As a consequence, usually more points will have to be used to obtain the desired result unless, of course, the tabulated values exactly represent a polynomial of degree at most one less than the number of points used. When interpolating in the end interval, it makes no difference whether the work progresses away from the unknown value or toward it. The former plan gives progressive values in the various stages that are closer to the final value, but the convergence is correspondingly slower. The final result is the same, within limits imposed by rounding.

In the extreme case of using the method for extrapolation to a point one equal argument-interval outside of the set of points, the method gives the same result as is obtained by differencing the tabulated values and extrapolating by assuming a constant (n − 1) the difference, where "n" is the number of tabulated values that have to be used to produce convergence.

SUMMATIONS OF X, Y, X^2, Y^2 AND 2XY
TWO-DIGIT AMOUNTS.

EXAMPLE:

X	and	Y
95		85
96		91
97		95
75		69
70		65
433		405

DECIMALS: For 10-column Marchant: Keyboard Dial 7 and 0; Upper Dial 7 and 0; Middle Dial 14, 7 and 0. Set Tab Keys 9 and 2.

OPERATIONS: *(1)* With Carriage in Position 9, enter X at left and Y at right of Keyboard Dial, around pre-set decimals. (Keyboard Dial shows: 095.0000085.)

(2) Enter X and Y in Multiplier Keyboard so that X appears at left and Y at right of Upper Dial, around pre-set decimals:

With Carriage in Position 9, enter X (95.) in Multiplier Keyboard. Shift Carriage to Position 2. (If your Marchant has "live" tab keys, simply touch Tab Key "2" to position Carriage.)

Enter Y (85.) in Multiplier Keyboard.

Upper Dial now shows: 095.0000085. and
Middle Dial shows: 9025.0016150.0007225.

NOTE: *Always read Upper Dial entries (X and Y) direct from Keyboard Dial instead of from original work.*

(3) Shift Carriage to Position 9. Change Keyboard Dial to second values of X and Y (096.0000091.). Reading from Keyboard Dial, multiply by this same number.

(4) Proceed as described above for subsequent X and Y values. Upon completion, Upper Dial shows ΣX, ΣY: 433.0000405.; Middle Dial shows ΣX^2, $\Sigma 2XY$, ΣY^2: 38175.0071502.0033517.

$$\Sigma XY = 71502 \times .5 = 35751$$

CHECK: Add the X's and Y's as they appear on the original work. If the amounts agree with Upper Dial readings, it is substantially proved that the X's and Y's were properly entered in the Keyboard Dial and were properly used as multipliers. This would not necessarily be true if there were two equal and offsetting errors, but it is usually satisfactory to regard the likelihood of such errors as too remote for consideration. It will be observed that this check is not reliable unless the multipliers are read from the Keyboard Dial.

NOTE: AN ELEVENTH UPPER DIAL PROVIDES CAPACITY FOR APPROXIMATELY 200 PAIRS OF AMOUNTS.

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SUMMATIONS OF X, Y, X^2, Y^2 AND XY

EXAMPLE:

X	and	Y
195		187
215		198
185		176
225		214
820		775

DECIMALS: Keyboard Dial 0 and 7, Upper Dial 0, Middle Dial 0 and 7. Set Tab Key 3.

OPERATIONS:

(1) With carriage in Position 3, enter first value of X at "0" Keyboard Dial decimal and first value of Y at "7" Keyboard Dial decimal. (Keyboard Dial reads 187.0000 195.). Multiply by first X (195.).

(2) Without clearing any dials, shift carriage to Position 3, change Keyboard Dial to secon d values of X and Y (198.0000 215.) and multiply by second value of X (215) copying from right of Keyboard Dial rather than from worksheet.

(3) Repeat (2) for each value of X and Y. At conclusion, Upper Dial shows ΣX (820.), Middle Dial shows ΣXY (159745.), and ΣX^2 (169100.)

Clear all dials and repeat process, this time entering X values at 7th Keyboard Dial decimal and Y values at "0" Keyboard Dial decimal and multiplying by Y values. At conclusion, Upper Dial shows ΣY (775.), Middle Dial show ΣXY (159745.), and ΣY^2 (150945.)

If summations of XY agree in the two computations it is a satisfactory check that all summations are correct.

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SUMMATIONS OF X, X^2 AND XY, OR Y, Y^2 AND XY
TWO-DIGIT AMOUNTS

X	and	Y
95		85
96		91
97		95
75		69
70		65
433		405

DECIMALS: Keyboard Dial 0 and 6, Upper Dial 3, Middle Dial 3 and 9. Set Tab Key 5.

OPERATIONS:

(1) With carriage in Position 5 enter first value of X at "0" Keyboard Dial decimal and first value of Y at "6" Keyboard Dial decimal. (Keyboard Dial reads 0085.000095.) Multiply by first X (95.).

(2) Without clearing any dials, shift carriage to Position 5, change Keyboard Dial to second values of X and Y (0091.000096.) and multiply by second value of X (96.).

(3) Repeat (2) for each value of X and Y. At conclusion, Upper Dial shows ΣX(433.), Middle Dial shows ΣXY (35751.) and ΣX^2(38175.).

If summations of Y, Y^2 and XY are needed follow above operations, enter Y values at "0" Keyboard Dial decimal, X values at "6" Keyboard Dial decimal and multiply by Y.

SUMMATIONS OF X^2 AND (UX^2), or X^3

A frequent summation required in Statistical Method and Least Squares is that of (UX^2) in which U equals 1/10, 1/100 or 1/1000 of X. It will be seen that this is similar to summing X^3 because U is the same as X except for the decimal point. (It is also often desired that summations of X^2 be made as a part of the same operation.)

(A) Summations of X^2 and (UX^2) for two-digit amounts:

EXAMPLE:

U	X	U	X
.65	*65*	*.74*	*74*
.46	*46*	*.89*	*89*

DECIMALS: Keyboard Dial 8 and 0; Upper Dial 1; Middle Dial 9 and 1.
Set Tab Keys 3 and 5.

OPERATIONS:

(1) With Carriage in Position 3, enter first X (65.) in Keyboard Dial at "0" decimal, and multiply by X (65.) around 1st Upper Dial decimal (reading X from right of Keyboard Dial, *not* from original work.)

First X^2 (4225.) appears in Middle Dial.

(2) Enter first U (.65) in Keyboard Dial at 8th decimal, with right hand digit reduced by "1", and fill in columns to right with 9's.

Keyboard Dial now reads: .64999999.

(Be sure, in entering U, to read it from right of Keyboard Dial, *not* from work sheet.)

(3) Shift Carriage to Position 5, and multiply by amount appearing in Middle Dial at 1st decimal (4225.).

Now first (UX^2) appears in Middle Dial at left (2746.25) and Middle Dial at right shows all ciphers.

(4) Clear-TAB Keyboard Dial only. Repeat steps 1, 2 and 3 for each succeeding pair.

Upon completion,

$\Sigma(UX^2)$ appears in Middle Dial at 9th decimal (14821.54)
$\Sigma(X^2) + \Sigma X$ appears in Upper Dial (20012)

(5) To subtract ΣX from Upper Dial, negatively multiply by ΣX (274, as separately totaled), leaving remainder in Upper Dial: ΣX^2 (19738).

(OVER)

NOTE: If directions have been followed, the work will be practically error-proof. However, there is no check of this except by obtaining $\Sigma(X^2)$ by accumulative multiplication. If it agrees with $\Sigma(X^2)$ as found in step 5, $\Sigma(UX^2)$ is substantially proved, provided the entries in step 2 were actually the same (less 1) as the X at right of the Keyboard Dial.

(B) Summations of X^2 and (UX^2) for Three-Digit Amounts.

This is done in a similar manner to that of Example A, except that a 10 column Marchant must be used.

The capacity is about 40 pairs of three-digit scores averaging 500. If the scores tend to concentrate at any other value, the capacity of the process would be affected correspondingly.

(C) Summations of (UX^2) without obtaining Summation of X^2.

By clearing Upper Dial after steps 2 and 3, $\Sigma(UX^2)$ will be obtained with a somewhat better proof of accuracy, particularly if the entry at left of Keyboard Dial in step 2 is made by copying it from the Upper Dial and checking it with the amount at right of Keyboard Dial, thus providing a "closing proof".

The fact that after step 3, Middle Dial at right shows all ciphers is proof that multiplication was actually made by X^2.

OUTLINE OF SUMMATION – WORK FOR LINEAR MULTIPLE CORRELATION

REMARKS: It is assumed that the computer is familiar with the customary method of obtaining the regression equations of multiple linear correlation in the form

$$X_1 = a + b_2X_2 + b_3X_3$$

or

$$X_1 = a + b_2X_2 + b_3X_3 + b_4X_4$$

for two and three variables, respectively. The calculating work principally comprises obtaining the required summations that become the coefficients of the terms of the normal equations.

These normal equations are then solved for the respective a and b values, preferably by the Crout Method (see Marchant Methods MM-434 B1 and MM-434 B2). It is recommended that the original scores be reduced to two-digit amounts in cases where they are larger. This may be done by either subtracting a fixed amount, or by dividing by 10, 100, etc., and rounding to a two-digit amount. The coefficients obtained by use of these modified scores are easily converted to ones suitable for use with unmodified scores.

It is also recommended that the original scores be used instead of their deviations from the point of averages. Though this adds slightly to the work of solving the normal equations, the time is believed to be more than offset by the elimination of obtaining deviations from the point of averages, as well as the time of converting the resulting regression equation into one that is usable with raw scores.

OUTLINE FOR CASE OF TWO INDEPENDENT VARIABLES:

The normal equations are

$$\text{I} \quad \Sigma(X_1) = aN + b_2\Sigma(X_2) + b_3\Sigma(X_3)$$

$$\text{II} \quad \Sigma(X_1X_2) = a\Sigma(X_2) + b_2\Sigma(X_2^2) + b_3\Sigma(X_2X_3)$$

$$\text{III} \quad \Sigma(X_1X_3) = a\Sigma(X_3) + b_2\Sigma(X_2X_3) + b_3\Sigma(X_3^2)$$

The summations are found as follows:

1st. According to Marchant Method MM–441 A1, find ΣX_2; ΣX_3; $\Sigma(X_2^2)$; $\Sigma(X_3^2)$; $\Sigma(2X_2X_3)$,

2nd. Enter in Keyboard Dial each X_2 at left and X_3 at right, and multiply by the corresponding X_1, producing by accumulation

$$\Sigma(X_1X_2) \quad \text{and} \quad \Sigma(X_1X_3)$$

The first step is self-checking if the procedure of Marchant Method MM–441 A1 is followed, so it need not be repeated for proof. The second step should be repeated. Sub-totals taken every 20 scores will aid in the proof.

(over)

OUTLINE FOR CASE OF THREE INDEPENDENT VARIABLES:

The normal equations are

$$\text{I} \quad \Sigma(X_1) = aN + b_2\Sigma(X_2) + b_3\Sigma(X_3) + b_4\Sigma(X_4)$$

$$\text{II} \quad \Sigma(X_1X_2) = a\Sigma(X_2) + b_2\Sigma(X_2^2) + b_3\Sigma(X_2X_3) + b_4\Sigma(X_2X_4)$$

$$\text{III} \quad \Sigma(X_1X_3) = a\Sigma(X_3) + b_2\Sigma(X_2X_3) + b_3\Sigma(X_3^2) + b_4\Sigma(X_3X_4)$$

$$\text{IV} \quad \Sigma(X_1X_4) = a\Sigma(X_4) + b_2\Sigma(X_2X_4) + b_3\Sigma(X_3X_4) + b_4\Sigma(X_4^2)$$

The summations are found as follows:

1st. According to Marchant Method MM-441 A1 ΣX_2; ΣX_3; $\Sigma(X_2^2)$; $\Sigma(X_3^2)$; $\Sigma(2X_2X_3)$,

2nd. Similarly find ΣX_1; ΣX_4; $\Sigma(X_1^2)$; $\Sigma(X_4^2)$; $\Sigma(2X_1X_4)$

3rd. Enter in Keyboard Dial each X_2 and X_3 and multiply by the corresponding X_1, producing

$$\Sigma(X_1X_2) \quad \text{and} \quad \Sigma(X_1X_3)$$

4th. Enter in Keyboard Dial each X_2 and X_3 and multiply by the corresponding X_4, producing

$$\Sigma(X_2X_4) \quad \text{and} \quad \Sigma(X_3X_4)$$

The first and second operations are self-checking if the procedure of Marchant Method MM-441 A1 is followed, so they need not be repeated for proof. The third and fourth operations, however, should be repeated.

SUMMATION OF X, XY AND XY^2

REMARKS: In certain cases of statistical computing, it is often desired to find $\sum x$*, $\sum xy$ and $\sum xy^2$, when given a large number of pairs of factors "x" and "y". This method is suitable for cases in which the factors have no more than two digits each.

EXAMPLE:

Given:

x	y
2	7
3	12
31	43
14	17
5	22

Find:

$\sum x = 55$

$\sum xy = 1731$

$\sum xy^2 = 64315$

DECIMALS: Keyboard Dial 5 and 0; Upper Dial 5 and 0; Middle Dial 10, 5 and 0.

OPERATIONS:

(1) Enter first "y" (7) in Keyboard Dial at 5th decimal, and multiply by first "x" (2) so it appears in Upper Dial around "0" decimal.

(2) Decrease right-hand figure of "y" in Keyboard Dial by "1", and enter all 9's in columns at right so that Keyboard Dial reads 6.99999.

Position Carriage and multiply around 5th Upper Dial decimal by "xy" (14) which appears directly below in Middle Dial around 5th decimal.

The Middle Dial should now show all ciphers at right of 10th decimal.

xy^2 (98) appears at 10th Middle Dial decimal, and
xy (14) appears at 5th Upper Dial decimal, but the amounts need not be separately noted.

(3) Clear Keyboard Dial only, and proceed as in Steps 1 and 2 for the remaining pairs of values.

$\sum xy^2$ appears at left of Middle Dial.
$\sum xy$ appears at left of Upper Dial.
$\sum x$ appears at right of Upper Dial.

NOTE: If $\sum x$ at right of Upper Dial equals the sum of the x values when they are separately added, it is substantially a proof that all x values have been correctly entered as multipliers. Any error of entry of xy as multiplier in Step 2 or the improper filling-in of 9's, is signalized by Middle Dial failing to clear after Step 2.

It will thus be seen that this process provides first-run accuracy control, except for values of "y". However, the likelihood of an error in setting "y" is remote because it appears in the Keyboard Dial and it is also separately noted in that dial when its right-hand figure is reduced by 1.

(*) *The symbol $\sum$ indicates "the summation of."*

SUMMATION OF FACTORS OF THE TYPE OF $\frac{A\ B}{K}$ *WHEN A, B, AND K ARE VARIABLE*

EXAMPLE: $\frac{546.32 \times 39.75}{176.32} + \frac{392.56 \times 78.62}{194.55} + \frac{425.87 \times 55.83}{187.72} = 408.46$

SETUP: Decimals: Keyboard Dial 2; Upper Dial 3; Middle Dial 5. Tab Key No. 5 depressed.

OPERATIONS:

(1) Enter in Keyboard Dial the first A (546.32), and multiply by the first B (39.75), around pre-set decimals.

(2) Clear-TAB Keyboard Dial only. Enter the first K (176.32) in Keyboard Dial as divisor. Touch Stop and Division Keys so that Upper Dial does not clear.

The first quotient, plus the first B, appears in Upper Dial (1962.91), but it is not separately noted.

(3) Enter in Keyboard Dial the second A (392.56). Enter second B (78.62) in Multiplier Keyboard.

(4) Clear-TAB Keyboard Dial only. Enter second K (194.55) as divisor. Touch Stop and Division Keys so that Upper Dial does not clear.

The sum of the quotients, as well as of the first and second B's appear in Upper Dial (400.171), but it is not separately noted.

(5) Repeat steps 3 and 4 for the third expression.
The sum of the quotients, as well as of the three B's appear in Upper Dial (582.659), but it is not separately noted.

(6) Clear Keyboard Dial. From a separately made summation, the B's = 174.20. (Sum the B's with Carriage shifted to extreme right, if desired.)

With Carriage in Position 6, touch Repeat-Neg.X Keys; negatively multiply by the sum of the B's.

The desired sum of the quotients (408.459) appears in Upper Dial.

NOTE: The above shows calculation with positive amounts. Similar calculations when there are mixed positive and negative amounts are easily handled. In such cases, perform all multiplications as if the factors were positive. If inspection shows that the quotient is negative, move Manual Counter Control toward the operator before touching Division Key. The subtraction of B's in step 6 should then be the sum of all of the B's considered as positive amounts, even though some of them may be negative.

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STATISTICAL METHOD
MEAN AND STANDARD DEVIATION – DATA GROUPED BY EQUAL CLASS INTERVALS

REMARKS: "Standard Deviation" is an index figure widely used in statistical analysis. It is the square root of the quotient obtained by dividing the sum of the squares of the deviation of each of a number of amounts from its arithmetic mean by the number of such amounts; thus

$$\sigma = \sqrt{\frac{\Sigma(x^2)}{N}}$$

in which σ = Standard Deviation (the Greek small letter Sigma)
x = individual deviations from arithmetic mean
N = total number of items

The Standard Deviation is computed with respect to either the X or Y axis.

This method relates to obtaining the Standard Deviation with respect to either the X or Y axis when the data are grouped by "equal" classes; that is to say, the data are sorted into groups, the X (or Y) spread of all of the classes being the same, and each class being successively of an equal higher amount than its predecessor. This is by far the most usual form in which problems involving Standard Deviation appear.

EXAMPLE: 221 average grades of students vary between marks of 59.0 and 93.8. They are grouped for convenience of calculating into class intervals of 2 points each, ranging from 58.0 to 100.0. The number of students whose average marks fall in each class interval is shown by tallies on the attached sheet. The total in each group, or "frequency", is shown in column f. The number of the class interval, reading from the bottom, is shown in column d. This column also designates the interval, or "deviation", of each group from the Arbitrary Origin (A.O.) of 59, which is the mid-point of the lowest group. The columns at the right are, respectively,

d^2 equals the square of amount in column d
d_1 equals d increased by "1"

${d_1}^2$ equals the square of amount in column d_1

For any given pattern of calculating, sheets are prepared in advance and mimeographed or printed with all amounts in columns d, d^2, d_1 and ${d_1}^2$ completed. The only part filled in by the operator is the tally marks and their totalizing in the column headed "f".

OUTLINE: In this method the actual deviations from the arithmetic mean are not obtained. The final result, however, is the Standard Deviation exactly as defined.

The Charlier Check of the summations is a part of the process.

(over)

PART I: SUMMATIONS OF f, fd, $f(d^2)$, fd_1, $f(d_1{}^2)$

DECIMALS: Keyboard Dial 8 and 0; Upper Dial 6; Middle Dial 14 and 6. Set Tab Key 8.

OPERATIONS: In all of these summations, the multipliers (f) are entered as if they were two-digit amounts; e.g., "3" is entered as "03", etc.

1) With carriage in Position 8, and without Keyboard Dial entries, multiply by the frequency (01) corresponding to the group at Arbitrary Origin (59).

   Note: This is done to record that there was 1 tally in class 60-58. As the midpoint is the Arbitrary Origin, there is no deviation, so the operation is one of multiplying the "0's" of column d and d^2 by the f of the group. Touch TAB BAR.

2) Enter in Keyboard Dial at 8th and "0" decimals respectively the amounts that are in columns d and d^2 for group 62-60 (1.0 and 1.0) and multiply by corresponding frequency (03). Touch TAB BAR.

3) Change Keyboard Dial to read the next greater amounts of d and d^2 (2.0 and 4.0) and multiply by corresponding frequency (15). Touch TAB BAR.

4) Change Keyboard Dial to read the next greater amounts of d and d^2 (3.0 and 9.0) and multiply by corresponding frequency (19). Touch TAB BAR.

5) Proceed as in Step 4 for all other groups.

   Sigma f (221) appears in Upper Dial.
   Sigma fd (1,429.0) and Sigma $f(d^2)$ (11,265.0) appear at left and right of Middle Dial.

6) Clear all dials and with carriage in Position 8 enter d_1 and $d_1{}^2$ (1.0 and 1.0) at 8th and "0" Keyboard Dial decimals respectively and multiply by corresponding frequency (01).

7) Change Keyboard Dial to read the next greater amounts of d_1 and $d_1{}^2$ (2.0 and 4.0) and multiply by corresponding frequency (03).

8) Proceed as in Step 7 for all other groups.

   Sigma f (221) appears in Upper Dial.

   Sigma fd_1 (1,650.0) and Sigma $f(d_1{}^2)$ (14,344.0) appear at left and right of Middle Dial.

(continued)

PART II: THE CHARLIER CHECK

In order to prove the correctness of the summations, they are checked as follows:

(a) Σfd (1429) + Σf (221) = Σfd_1 (1650)
(b) $\Sigma f(d^2)$ (11265) + 2 Σfd (2858) + Σf (221) = $\Sigma f(d_1{}^2)$ (14344)

Any failure to reconcile will require a re-run of the summations.

1) With carriage in Position 1, enter Sigma f (221) in Keyboard Dial at "0" decimal. Touch ADD BAR.

2) Similarly enter Sigma fd (1429) and multiply by "1".
Sigma fd_1 (1650) appears in Middle Dial.

3) Clear Upper and Middle Dials and multiply by 2.

4) Change Keyboard to read Sigma $f(d^2)$ (11265) and add.

5) Enter Sigma f (221) and add.
Sigma $f(d_1{}^2)$ (14344) appears in Middle Dial.

PART III: STANDARD DEVIATION IN CLASS INTERVAL UNITS

Much work of this type merely requires that the Standard Deviation be obtained in terms of Class Intervals; that is to say, each Class Interval (in this case 2) is regarded as the unit of measurement. Obviously, the Standard Deviation in terms of the units from which the Class Intervals are obtained equals the "Standard Deviation in terms of Class Intervals" multiplied by the "Number of Units in a Class Interval".

The formula is:

$$\text{Standard Deviation in terms of Class Intervals} = \frac{\sqrt{(\text{Sigma } f)\ (\text{Sigma } f(d)^2) - (\text{Sigma } fd)^2}}{\text{Sigma } f}$$

$$\text{or, in this example,} \quad = \frac{\sqrt{221 \times 11265 - 1429^2}}{221}$$

DECIMALS: Keyboard Dial 8, 5, and 2; Upper Dial 5 and 1; Middle Dial 13, 10 and 7.

1) Enter Sigma $f(d^2)$ (11265) in Keyboard Dial at decimal 2 and multiply by (Sigma f) (221) at Upper Dial decimal 5.

2) Clear Upper and Keyboard Dials. Move Manual Counter Control toward the operator.
Enter Sigma fd (1429) in Keyboard Dial at decimal 2, and negatively multiply by Sigma fd (1429) at Upper Dial decimal 6. Move Manual Counter Control away from operator.
The amount under radical sign (447524) appears in Middle Dial at decimal 7.

3) Refer to Marchant Table of Square Root Divisors, No. 82, $\sqrt{447524}$ = 668.97 +

4) Enter 668.97 in Middle Dial at decimal 10.

5) Shift carriage to Position 6. Change Keyboard Dial to read Sigma f (221) at decimal 5 and divide.

6) Standard Deviation in Class Interval Units is 3.0270 (see Note C).

(over)

PART IV: STANDARD DEVIATION IN UNITS FROM WHICH CLASS INTERVALS ARE DERIVED

This is obviously the result of Part III, Step 6, multiplied by the number of units per Class Interval, or, in this case,

$$2 \times 3.0270 \text{ or } 6.0540 \text{ (See Note C)}$$

PART V: ARITHMETIC MEAN

This is $\frac{\text{Sigma fd}}{\text{Sigma f}} \times$ Number of Units per Class Interval + A.O., in which A.O. is "Arbitrary Origin", or,

in this case, $\left(\frac{1429}{221} \times 2\right) + 59$, or 71.9321 (see Note E).

NOTES

A. The work has been subdivided into parts in order to clarify the individual parts of the work.

B. If it is desired to obtain directly the Standard Deviation in Units from which the Class Intervals are derived (Part IV), the divisor in Step 6 of Part III can be Sigma f divided by the number of units that comprise a class interval.

C. The method is based upon the assumption that all scores are concentrated at the mid point of each class interval. It is probable, however, that they are continuously distributed throughout the interval. The inherent error from this assumption causes the Standard Deviation to be slightly less than the values shown in Parts III and IV. For this reason it is recommended that the Standard Deviations be rounded downward to not more than 3 significant figures, thus,

For Part III to 3.02
For Part IV to 6.05

Most textbooks dealing with this subject show how to correct the Standard Deviations obtained by this method for continuous distribution in an interval and tapering frequencies.

D. "Sigma f" is also referred to as "N" in many formulas. It is the total number of individual scores.

E. After the division, the addition of 59 may be made in the Upper Dial by multiplying by 59. This causes Upper Dial to provide a direct reading of 71.9321

F. The distribution shown herewith would be described as
Mean 71.932 with Standard Deviation of 6.05

(continued)

WORK SHEET FOR MARCHANT METHOD
MEAN AND STANDARD DEVIATION

Data Grouped by Equal Class Intervals

Computer ____________________
Checker ____________________
Class Interval *2*
Arbitrary Origin *59*

Name of Report ____________________
Description ____________________
Mean *71.932*
Standard Deviation *6.05*

GROUP RANGE	SCORE TALLIES	f	d	d^2	d_1	d_1^2
100 - 98			20	400	21	441
98 - 96			19	361	20	400
96 - 94			18	324	19	361
94 - 92	/	*1*	17	289	18	324
92 - 90		*0*	16	256	17	289
90 - 88	////	*4*	15	225	16	256
88 - 86	///	*3*	14	196	15	225
86 - 84	/	*1*	13	169	14	196
84 - 82	////	*4*	12	144	13	169
82 - 80	~~////~~ /	*6*	11	121	12	144
80 - 78	~~////~~ ~~////~~ /	*11*	10	100	11	121
78 - 76	~~////~~ ~~////~~ ~~////~~ ~~////~~ /	*21*	9	81	10	100
76 - 74	~~////~~ ~~////~~ ~~////~~ ////	*19*	8	64	9	81
74 - 72	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ///	*28*	7	49	8	64
72 - 70	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~	*40*	6	36	7	49
70 - 68	~~////~~ ~~////~~ ~~////~~ ~~////~~ ~~////~~ /	*26*	5	25	6	36
68 - 66	~~////~~ ~~////~~ ~~////~~ ////	*19*	4	16	5	25
66 - 64	~~////~~ ~~////~~ ~~////~~ ////	*19*	3	9	4	16
64 - 62	~~////~~ ~~////~~ ~~////~~	*15*	2	4	3	9
62 - 60	///	*3*	1	1	2	4
60 - 58	/	*1*	0	0	1	1

Σf or N *221*

NOTE: All items that are printed on the above form appear in regular type. Only the tallies and the italicized figures are entered by the computer.

STATISTICAL METHOD

LINEAR "LEAST SQUARES" LINE OF REGRESSION

AND COEFFICIENT OF REGRESSION

REMARKS: *It is often desired to know the equation of the straight line that best represents a scatter diagram according to the principle of "least squares;" e.g., the line is such that the sum of the squares of the deviations of observed values of Y from the corresponding values of Y on the line will be a minimum. If this line be represented by the equation $Y = a + bX$; the coefficient "b" represents the slope of the line or defines its angle with reference to the X axis ($\tan^{-1} b$) and "a" represents the Y intercept.*

The coefficient "b" of this least squares line of regression is sometimes referred to as the "Coefficient of Regression."

EXAMPLE: Given the observed values of X and Y of Marchant Method MM444 A, find the linear Coefficient of Regression and also the equation of the line assuming the regression to be linear. To avoid reference to MM444 A, it is noted here that:

Sigma X equals 433	Sigma X^2 equals 38175
Sigma Y equals 405	Sigma Y^2 equals 33517
Sigma XY equals 35751	N equals 5

OUTLINE: The equation for the line of regression is determined by the solution for "a" and "b" in the two equations below:

(I) Sigma Y = Na + b Sigma X
(II) Sigma XY = a Sigma X + b Sigma X^2

Solving these equations for "b" and transforming so the equation is best suited for machine calculation, we have:

(III) $$b = \frac{\text{N Sigma XY} - \text{Sigma X Sigma Y}}{\text{N Sigma } X^2 - (\text{Sigma X})^2}$$

(IV) $$a = \frac{\text{Sigma Y} - \text{b Sigma X}}{N}$$

DECIMALS: Upper Dial 5. Middle Dial 10, Keyboard Dial 5.
Use 10 column Marchant.

To Compute "b"

OPERATIONS:

(1) Enter in Keyboard Dial Sigma (X^2) (38175) and multiply by (5).

(2) Move Manual Upper Dial Control toward operator, clear Upper and Keyboard Dials, enter in Keyboard Dial Sigma X (433) and negatively multiply by Sigma X (433).

Middle Dial shows N Sigma X^2 – (Sigma X)^2 (3386) Copy to report.

(over)

(3) Move Manual Upper Dial Control away from operator. Clear dials, enter in Keyboard Dial Sigma XY (35751) and multiply by N (5).

(4) Move Manual Upper Dial Control toward operator, clear Upper and Keyboard Dials, enter in Keyboard Dial Sigma X (433) and negatively multiply by Sigma Y (405).

Middle *Dial shows N Sigma XY – Sigma X Sigma Y (3390).*

(5) Move Manual Upper Dial Control away from operator, clear Keyboard and Upper Dials, enter in Keyboard Dial the amount copied to report from Step 2 (3386) and divide.

Coefficient of Linear Regression "b" (1.0012) appears in Upper Dial.

To Compute "a"

(a) Enter in Keyboard Dial Sigma Y (405) and multiply by "1."

(b) Clear Keyboard and Upper Dials, enter in Keyboard Dial Sigma X (433), move Manual Upper Dial Control toward the operator and negatively multiply by "b" (1.0012). If the amount in Middle Dial were positive, procedure would be as in Step "1c" below.

(1c) Move Manual Upper Dial Control away from operator, clear Upper and Keyboard dials, enter N (5) in Keyboard Dial, and divide.

Coefficient "a," the Y intercept, appears in Upper Dial.

In this case the amount in Middle Dial is a negative number, i.e., (...99971.4804). This amount may be evaluated as a positive number and then divided by N, or advantage may be taken of the ability of the Marchant to divide a negative number directly; thus, as in this case, proceeding from Step "b";
Move Manual Upper Dial Control away from the operator.

(2c) The Middle Dial after Step "b" reads (...99971.4804). Clear Upper and Keyboard Dials, enter N (5) in Keyboard Dial . Shift to Position 7 (so the left digit of the divisor is directly below the first digit to the right of the 9's), multiply by anything that clears the 9's (in this case multiplying by "1" will do so).

(d) Move Manual Upper Dial Control toward the operator and divide.* Coefficient "a" (5.704) appears in Upper Dial, but it is to be written as – 5.704, because the Upper Dial was producing negative quotients while "a" was developed.

The equation for the Linear "Least Squares" Line of Regression is thus:

$$Y = -5.074 + 1.0012X$$

(*) *Depress Division Key in the manner that prevents Upper Dial clearance.*